A note on a theorem of Armand Borel

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A NOTE ON A THEOREM OF ARMAND BOREL

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The theorem under discussion is the one which yields the cohomology of the classifying space of a Lie group. Let E be a canonical spectral algebra for cohomology over a field K with trivial E_∞ term, and let $B = \sum_p E_2^{p,0}$, $F = \sum_q E_2^{0,q}$ (the algebras corresponding to the cohomologies of base and fibre of a fibre space).

THEOREM 1. *If* $F = \Lambda(x_1, \ldots, x_m)$, *an exterior algebra on homogeneous elements of odd degree, then*

(*a*) *homogeneous transgressive elements* $y_1, \ldots, y_m$ *can be chosen such that*

$$F = \Lambda(y_1, \ldots, y_m), \quad \text{and} \quad \text{degree } y_i = \text{degree } x_i;$$

(*b*) $B = K[z_1, \ldots, z_m]$, *a polynomial ring, where* z_i *is an arbitrary image of* y_i *under transgression.*

Borel gives an intricate proof of this in his thesis ((1), Theorem 13·1, p. 157). He points out that the theorem is essentially one of uniqueness, and the proof given here is based on this remark. I prove (*b*) by assuming (*a*) and mapping a suitably manufactured spectral algebra into E; the map turns out to be an isomorphism by the comparison theorem (2). I do not know whether (*a*) can also be proved by this method.

The technique is applicable to a variety of spectral sequence arguments where the answer can be guessed and uniqueness has to be proved. For example it extends very simply to the case:

THEOREM 2. *If K is of characteristic 2, and if F admits of a simple system of homogeneous transgressive generators of positive degree,* $F = \Delta(y_1, \ldots, y_m)$, *then* $B = K[z_1, \ldots, z_m]$, *where* z_i *is an arbitrary image of* y_i *under transgression.*

Borel states this ((1), Theorem 16·1); the proof is given below.

The *tensor product* $E = E' \otimes E''$ of two given spectral algebras E' and E'' is defined as follows:

Let $E_r = E'_r \otimes E''_r$, the tensor product of two bigraded associative skew-commutative algebras over K, with derivation, yielding another such. Since we are working over a field, $H(E'_r \otimes E''_r) = H(E'_r) \otimes H(E''_r)$, and so the formula $E_{r+1} = H(E_r)$ is satisfied. The properties are immediate:

(i) If E' and E'' have trivial ∞-terms, so has E.

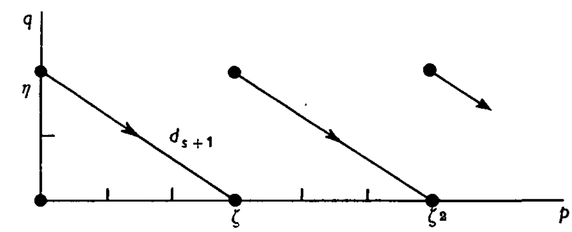
(ii) $F = F' \otimes F''$, $B = B' \otimes B''$.

(iii) The associativity of $\otimes$ for spectral algebras follows from that for algebras.

(iv) Two homomorphisms $f': \bar{E}' \to E'$ and $f'': \bar{E}'' \to E''$ induce a homomorphism $f' \otimes f'': \bar{E}' \otimes \bar{E}'' \to E' \otimes E''$.

(v) The product in E induces a natural homomorphism $E \otimes \ldots \otimes E \to E$.

Definition of $E(s)$, an *elementary* spectral algebra over K of odd degree s. Let $F(s) = \Lambda(\eta)$, η of degree s, $B(s) = K[\zeta]$, ζ of degree $s+1$, and let $E(s)_2 = B(s) \otimes F(s)$. Therefore if K is not of characteristic 2, the algebra $E(s)_2$ is freely generated by ζ and η, the multiplicative order of η being 2 since s is odd. If K is of characteristic 2, then $E(s)_2$ is generated by ζ and η, with the one relation $\eta^2 = 0$. Let $d_r = 0$, $r = 2, 3, \ldots, s$. Therefore $E(s)_r = E(s)_2, r = 2, 3, \ldots, s+1$. Let $d^{0,s}_{s+1}(\eta) = \zeta$, (the transgression). Therefore $d_{s+1}(\zeta^k \otimes 1) = 0$ and $d_{s+1}(\zeta^k \otimes \eta) = \zeta^{k+1} \otimes 1$. Consequently, for $r > s+1$, $d_r = 0$ and $E(s)_r = E(s)_\infty =$ trivial, the only non-zero term in the bigrading being $E(s)^{0,0}_\infty = K$.



LEMMA. *If y is a transgressive element of odd degree s in a spectral algebra E such that $y^2 = 0$, and if z is some image of y in B under transgression, then there is a unique homomorphism $f: E(s) \to E$ such that $f\eta = y$ and $f\zeta = z$.*

Proof. We have to define for each r an algebra homomorphism $f_r: E(s)_r \to E_r$ such that f_r commutes with d_r and induces f_{r+1}. For $r > s+1$ the task is trivial since $E(s)_r$ is trivial.

Suppose $r \leqslant s+1$; let $\kappa_r z$ be the image of z under the epimorphism $\kappa_r: E_2^{s+1,0} \to E_r^{s+1,0}$, and define $f_r \eta = y$, $f_r \zeta = \kappa_r z$. This determines f_r uniquely since $E(s)$ is generated by ζ and η. Moreover, f_r is an algebra homomorphism since if K does not have characteristic 2 then $E(s)_r$ is freely generated, and if K has characteristic 2 the one relation $\eta^2 = 0$ in $E(s)_r$ is echoed by $y^2 = 0$ in E_r (this being the purpose of putting $y^2 = 0$ in the hypothesis). Now $\kappa_r z$ is a d_r-cocycle, and since y is transgressive, y is also a d_r-cocycle for $r = 2, 3, \ldots, s$. Therefore for these values of r commutativity is trivial, and f_r induces f_{r+1}. In the case $r = s+1$ the transgression $d^{0,s}_{s+1}$ maps η and ζ in $E(s)_{s+1}$ and y to $\kappa_{s+1} z$ in E_{s+1}. Therefore f_{s+1} commutes with d_{s+1} on the generators of, and so on the whole of, $E(s)_{s+1}$, and induces (the trivial) f_{s+2}. The uniqueness of f follows from that of f_2.

Proof of Theorem 1 (*b*). We assume (*a*). We are given E with $F = \Lambda(y_1, \ldots, y_m)$, where y_i is transgressive of odd degree s_i, say. We are also given for each i some image z_i of y_i in B under transgression. Let

$$\bar{E} = E(s_1) \otimes \ldots \otimes E(s_m).$$

Then

$$\bar{F} = \Lambda(\eta_1) \otimes \ldots \otimes \Lambda(\eta_m) = \Lambda(\eta_1, \ldots, \eta_m),$$

$$\bar{B} = K[\zeta_1] \otimes \ldots \otimes K[\zeta_m] = K[\zeta_1, \ldots \zeta_m].$$

Since $y_i^2 = 0$, the lemma gives a homomorphism $f^i: E(s_i) \to E$, mapping η_i to y_i and ζ_i to z_i, and hence a homomorphism $f: \bar{E} \to E$ by the composition

$$E(s_1) \otimes \ldots \otimes E(s_m) \xrightarrow{f^1 \otimes \ldots \otimes f^m} E \otimes \ldots \otimes E \xrightarrow{\text{product}} E.$$

By construction $f:\bar{F} \cong F$. Therefore $f: \bar{B} \cong B$, qua graded groups, by the comparison theorem ((2), Dual corollary). But f is an algebra homomorphism, so that $f: \bar{B} \cong B$ is an algebra isomorphism. Consequently $B = K[z_1, \ldots, z_m]$.

Proof of Theorem 2. We recall that $F = \Delta(y_1, \ldots, y_m)$ means that the monomials $y_{i_1} y_{i_2} \cdots y_{i_k}$, $i_1 < i_2 < \ldots < i_k$ $(k = 1, 2, \ldots, m)$, together with the unit element form an additive base for the vector space F over K. This is more general than an exterior algebra, since it may happen that $y_i^2 \neq 0$, as, for example, in the cohomology ring modulo 2 of the rotation group $R(3)$.

Since K is of characteristic 2 we may define elementary spectral algebras of even degree in exactly the same way as those of odd degree. For each i there is as before a spectral *sequence* homomorphism $f^i: E(s_i) \to E$, mapping η_i to y_i and ζ_i to z_i, only this time it is not strictly an *algebra* homomorphism, because, although $\eta_i^2 = 0$, we may have $(f^i \eta_i)^2 = y_i^2 \neq 0$. However, $f|B(s_i)$ *is* an algebra homomorphism since $B(s_i) = K[\zeta_i]$ is freely generated by ζ_i. The construction of f is as before. Then $f:\bar{F} \cong F$, qua additive structure only, and so $f: \bar{B} \cong B$, qua additive structure, since the comparison theorem depends only on the additive structure. But $f|\bar{B}$ is an algebra homomorphism, so that $f: \bar{B} \cong B$ is an algebra isomorphism.

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