ON THE POLYHEDRAL SCHOENFLIES THEOREM

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In this note we observe a relationship between the polyhedral Schoenflies problem and the question of whether the double suspension M^5 of a Poincaré manifold is the 5-sphere. In particular we show that if $M^5 = S^5$, then a polyhedral embedding of S^{n-1} in S^n must be very "nice" if the Schoenflies theorem is to hold.

DEFINITION 1. Let Δ^n be the standard n-simplex and $\dot{\Delta}^n$ be its boundary. A finite simplicial complex K is a *combinatorial n-sphere* if there exists a piecewise linear homeomorphism $h: K \to \dot{\Delta}^{n+1}$.

DEFINITION 2. An embedding $S^{n-1} \subset S^n$ is *nice* if there is a simplicial decomposition of S^n such that S^{n-1} is a subcomplex and both S^{n-1} and S^n are combinatorial spheres.

We have the following theorem (see [**5**]):

THEOREM 1. *If the embedding $S^{n-1} \subset S^n$ is nice, then the Schoenflies theorem holds; i.e., $S^n - S^{n-1}$ consists of two disjoint n-cells.*

DEFINITION 3. An embedding $S^{n-1} \subset S^n$ is of *type* I if S^n can be represented as a combinatorial n-sphere with S^{n-1} a subcomplex. An embedding is of *type* II if there is a simplicial decomposition of S^n such that S^{n-1} is a subcomplex which is a combinatorial $(n-1)$-sphere.

We construct a definite Poincaré manifold M^3 in S^4. Let P be the 2-polyhedron obtained by attaching the boundaries of two disks to two oriented curves a and b (with one common point) according to the formulae $a^{-2}bab = 1$, $b^{-5}abab = 1$. Then $\pi_1(P)$ has the presentation $\{a, b \mid a^{-2}bae = 1, b^{-5}abab = 1\}$, and Newman [**4**] has shown that $\pi_1(P) \neq 0$. Now P can be embedded in S^4 as a subcomplex (see [**2**; **4**]) with S^4 decomposed as a combinatorial 4-sphere. Then the boundary M^3 of a nice neighborhood of P is a Poincaré manifold [**2**]. It follows that the double suspension M^5 of M^3 is a subcomplex of the combinatorial 6-sphere S^6. This is used in Theorem 3.

We note that if M^5 is locally euclidean, then $M^5 = S^5$. For Mazur [**3**] has proved that if X is a finite polyhedron and the cone $C(X)$ is locally k-euclidean at the cone point, then $C(X) - X = E^k$. Now M^5 is the suspension of the single suspension M^4 of M^3 and the suspension with one suspension point removed is just $C(M^4) - M^4$. If this

Presented to the Society, September 2, 1960; received by the editors February 5, 1960.

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is E^5, then the suspension is just the 1-point compactification of E^5, namely S^5.

THEOREM 2. *If $M^5 = S^5$, then the Schoenflies theorem fails for embeddings of type* II *with $n = 5$.*

PROOF. Let σ be a 3-simplex of M^3 with boundary β. Then the double suspension of β is a combinatorial 4-sphere S^4 in $M^5 = S^5$. But $\pi_1(M^3 - \sigma) = \pi_1(M^3) \neq 0$ and one complementary domain of S^4 in S^5 is just $(M^3 - \sigma) \times I$, which is not simply connected.

THEOREM 3. *If $M^5 = S^5$, then the Schoenflies theorem fails for embeddings of type* I *with $n = 6$.*

PROOF. By the construction of $M^3 \subset S^4$ given above, we have that $M^5 = S^5 \subset S^6$ is an embedding of type I with $n = 6$. Let D^3 be the complementary domain of M^3 in S^4 which contains P. By projecting from suspension points we can get deformation retractions of a complementary domain D^5 (of $S^5 \subset S^6$) onto D^4 (of $M^4 \subset S^5$) onto D^3. Hence D^5 is not simply connected and therefore is not a cell.

REMARK. Since it seems difficult to prove that $M^5 \neq S^5$, it must be hard to show that embeddings of types I and II satisfy the simple condition $S^{n-1} \times I \subset S^n$ which Morton Brown [1] has shown is a necessary and sufficient hypothesis for the Schoenflies theorem.

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