

On the Polyhedral Schoenflies Theorem Author(s): M. L. Curtis and E. C. Zeeman Source: Proceedings of the American Mathematical Society, Vol. 11, No. 6 (Dec., 1960), pp. 888- 889 Published by: [American Mathematical Society](http://www.jstor.org/action/showPublisher?publisherCode=ams) Stable URL: [http://www.jstor.org/stable/2034432](http://www.jstor.org/stable/2034432?origin=JSTOR-pdf) Accessed: 24/09/2014 22:16

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ON THE POLYHEDRAL SCHOENFLIES THEOREM

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In this note we observe a relationship between the polyhedral Schoenflies problem and the question of whether the double suspension M" of a Poincare manifold is the 5-sphere. In particular we show that if $M^5 = S^5$, then a polyhedral embedding of S^{n-1} in S^n must be **very "nice" if the Schoenflies theorem is to hold.**

DEFINITION 1. Let Δ^n be the standard *n*-simplex and Δ^n be its **boundary. A finite simplicial complex K is a combinatorial n-sphere if there exists a piecewise linear homeomorphism** $h: K \rightarrow \Delta^{n+1}$ **.**

DEFINITION 2. An embedding $S^{n-1} \subset S^n$ is *nice* if there is a simplicial decomposition of $Sⁿ$ such that $Sⁿ⁻¹$ is a subcomplex and both $Sⁿ⁻¹$ and **Sn are combinatorial spheres.**

We have the following theorem (see [5]):

THEOREM 1. If the embedding $S^{n-1} \subset S^n$ is nice, then the Schoenflies *theorem holds; i.e.,* $Sⁿ - Sⁿ⁻¹$ consists of two disjoint n-cells.

DEFINITION 3. An embedding $S^{n-1} \subset S^n$ is of type I if S^n can be **represented as a combinatorial** *n***-sphere with** S^{n-1} **a subcomplex. An** embedding is of type II if there is a simplicial decomposition of $Sⁿ$ such that S^{n-1} is a subcomplex which is a combinatorial $(n-1)$ **sphere.**

We construct a definite Poincaré manifold M^3 in S^4 . Let P be the **2-polyhedron obtained by attaching the boundaries of two disks to two oriented curves a and b (with one common point) according to** the formulae $a^{-2}bab=1$, $b^{-5}abab=1$. Then $\pi_1(P)$ has the presentation $\{a, b\}a^{-2}bae=1, b^{-5}abab=1\}$, and Newman [4] has shown that $\pi_1(P) \neq 0$. Now P can be embedded in S^4 as a subcomplex (see [2; **4]) with S4 decomposed as a combinatorial 4-sphere. Then the boundary M3 of a nice neighborhood of P is a Poincare manifold [2]. It** follows that the double suspension M^5 of M^3 is a subcomplex of the combinatorial 6-sphere S^6 . This is used in Theorem 3.

We note that if M^5 is locally euclidean, then $M^5 = S^5$. For Mazur **[3] has proved that if X is a finite polyhedron and the cone C(X) is locally k-euclidean at the cone point, then** $C(X) - X = E^k$ **. Now** M^5 is the suspension of the single suspension M^4 of M^3 and the suspen**sion with one suspension point removed is just** $C(M^4) - M^4$ **. If this**

Presented to the Society, September 2, 1960; received by the editors February 5, 1960.

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is E^5 , then the suspension is just the 1-point compactification of E^5 , namely S^5 .

THEOREM 2. If $M^5 = S^5$, then the Schoenflies theorem fails for embeddings of type \prod with $n=5$.

PROOF. Let σ be a 3-simplex of M^3 with boundary β . Then the **double suspension of** β **is a combinatorial 4-sphere** S^4 **in** $M^5 = S^5$ **. But** $\pi_1(M^3-\sigma) = \pi_1(M^3) \neq 0$ and one complementary domain of S^4 in S^5 is just $(M^3-\sigma)\times I$, which is not simply connected.

THEOREM 3. If $M^5 = S^5$, then the Schoenflies theorem fails for em*beddings of type* I with $n = 6$.

PROOF. By the construction of $M^3\subset S^4$ given above, we have that $M^5 = S^5 \subset S^6$ is an embedding of type I with $n = 6$. Let D^3 be the complementary domain of M^3 in S^4 which contains P. By projecting **from suspension points we can get deformation retractions of a com**plementary domain $D^{\mathfrak{s}}$ (of $S^{\mathfrak{s}} \subset S^{\mathfrak{s}}$) onto $D^{\mathfrak{s}}$ (of $M^{\mathfrak{s}} \subset S^{\mathfrak{s}}$) onto $D^{\mathfrak{s}}$. Hence D^{δ} is not simply connected and therefore is not a cell.

REMARK. Since it seems difficult to prove that $M^5 \neq S^5$, it must be **hard to show that embeddings of types I and II satisfy the simple** condition $S^{n-1} \times I \subset S^n$ which Morton Brown [1] has shown is a **necessary and sufficient hypothesis for the Schoenflies theorem.**

REFERENCES

1. Morton Brown, Outline of a proof of the generalised Schoenflies theorem, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 74-76.

2. M. L. Curtis and R. L. Wilder, The existence of certain types of manifolds, Trans. Amer. Math. Soc. vol. 91 (1959) pp. 152-160.

3. Barry Mazur, On embeddings of spheres, Bull. Amer. Math. Soc. vol. 65 (1959) pp. 59-65.

4. M. H. A. Newman, Boundaries of ULC sets in Euclidean space, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 193-196.

5. , The division of Euclidean space by spheres, Proc. Roy. Soc. London, to appear.

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