UNKNOTTING SPHERES IN FIVE DIMENSIONS

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Given a semi-linear embedding of S^2 in euclidean 5-space, we show that it is unknotted.

Join it up to a vertex V in general position. If the cone VS^2 is non-singular we are finished. Otherwise, for dimensional reasons, there are at most a finite number of singularities, where just two points of S^2 are collinear with V. Let's have V away on one side, so that at each singularity we can call one point "near" and the other point "far." Now separate the near and far points by an equator S^1, so that all the near points lie in the northern hemisphere A, and all the far points lie in the southern hemisphere B.

Let $\hat{S}^2$ be the sphere $VS^1 \cup B$. Then $\hat{S}^2$ is equivalent to S^2, because they differ by the boundary of the ball VA, whose interior does not meet them. But $\hat{S}^2$ is unknotted because it bounds, and does not meet the interior of, the ball VB. Hence S^2 is unknotted.

REMARK 1. The argument generalizes to unknotting S^n in k-space, $k > (3/2)(n+1)$.

REMARK 2. I suspect that S^3 knots in 6-space (the first unsolved case), because the near set can link the far set.

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