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UNKNOTTING SPHERES

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For which dimensions n and k can a combinatorial n-sphere S^n, tamely embedded in euclidean k-space E^k, be knotted?

At the lower end of the scale, when $k = n + 2$, we can construct knots by a generalisation of Artin's method [2] of spinning 1-dimensional knots. This is the lowest[1] dimension in which knots can exist, and the only one in which we know they do exist.

At the upper end of the scale, it is easy to see that any S^n in E^k is unknotted if $k \geqq 2n + 2$; one only has to put one's eye in general position and glance at it[2]. We improve this result down to about halfway, and show in Theorem 2 that any S^n is unknotted in E^k if $k > (3/2)(n + 1)$. Nothing is known about the intervening dimensions $n + 2 < k \leqq (3/2)(n + 1)$. The first case affected by our theorem is the unknotting of S^2 in E^5 (see [8]). The first unsolved case is now S^3 in E^6.

The theorem also focuses an essential difference between linking and knotting, for it is easy to link two n-spheres in dimensions as high as E^{2n+1} (see [7]).

The first half of the paper is devoted to establishing in Theorem 1 the equivalence of five different criteria of unknottedness. In the second half we use one of these criteria, equivalence by cellular moves to the boundary of a ball, to prove the unknotting theorem, Theorem 2.

Definitions

Let Δ^n be a standard n-simplex. Recall that a *combinatorial n-ball* (or *combinatorial n-sphere*) is a finite simplicial complex piecewise linearly homeomorphic to Δ^n (or $\dot{\Delta}^{n+1}$). A *combinatorial n-manifold M* is a finite simplicial complex such that the link of each vertex is either a combinatorial $(n-1)$-sphere, if the vertex is in the interior of M, or a combinatorial $(n-1)$-ball, if the vertex is in the boundary $\dot{M}$ of M. Call M *closed* if $\dot{M}$ is empty. In the future whenever we say sphere, ball, or manifold, we shall always mean a combinatorial sphere, combinatorial ball, or combinatorial manifold, embedded rectilinearly in some euclidean

[1] When $k = n + 1$ the generalised Schönflies Theorem [3] is applicable, because S^n is tamely embedded, and so S^n is unknotted in the sense that it bounds a topological ball. It is not known yet whether it bounds a combinatorial ball, i.e., is unknotted in our sense.

[2] This is made rigorous by Theorem 1.

space. Define an *equatorial decomposition* of an n-sphere S^n to be a pair of n-balls B_1, B_2 such that $B_1 + B_2 = S^n$, $B_1 \cap B_2 = S^{n-1}$.

Definition of unknottedness

If S^n is embedded in E^k (or S^k), $k > n$, define S^n to be *unknotted* if there is a piecewise linear homeomorphism of E^k (or S^k) onto itself throwing S^n onto the boundary of an $(n+1)$-simplex. If M is an n-manifold embedded in E^k, $k > n$, define M to be *locally unknotted* if each point in the interior of M has an arbitrarily small k-ball neighbourhood N in E^k, such that $M \cap \dot{N}$ is an $(n-1)$-sphere that is unknotted in the $(k-1)$-sphere $\dot{N}$. For example a simplex is locally unknotted, and the boundary of a simplex is locally unknotted.

Both unknottedness and local unknottedness are properties which are invariant under piecewise linear homeomorphism. Therefore if S^n is unknotted, then it is locally unknotted. On the other hand consider the suspension in E^4 of a knotted S^1 in E^3; this is a 2-sphere which is locally-knotted at the suspension points, and consequently is also knotted. The spun knots of Artin [2] are examples of spheres which are knotted, but locally unknotted.

Simplicial moves

A second approach to the knotting of spheres is given by generalising to n-dimensions the classical notion of equivalence of polygonal knots by simplicial moves across triangles.

We say that two n-manifolds M_1, M_2 embedded in E^k, $k > n$, *differ by a simplicial move*[3] across the $(n+1)$-simplex A, if the interior of A does not meet $|M_1|, |M_2|$, and A is the join $A = BC$ of simplexes $B \in M_1$, $C \in M_2$, such that $\overline{M_1 - M_2} = B\dot{C}$, $\overline{M_2 - M_1} = \dot{B}C$. We denote the simplicial move from M_1 to M_2 by the symbol (B, C). Two n-manifolds M_1, M_r embedded in E^k, $k > n$, are *equivalent by simplicial moves* if there is a sequence $M_1, M_2, \cdots, M_r$ of n-manifolds in E^k, each one either differing from the next by a simplicial move, or else having the same underlying polyhedron as the next.

Cellular moves

In the proof of Theorem 2 we shall have to use a more complicated type

[3] If two manifolds differ by a simplicial move, then they are combinatorially equivalent by a Newman [4] move of type 3; however, the concept of differing by a simplicial move includes more than just a Newman move, for it also embodies the embeddings in E^k, with the interior of A not meeting $|M_1|, |M_2|$.

of move, across a ball rather than a simplex. We say that two n-manifolds M_1, M_2 embedded in E^k, $k > n$, *differ by a cellular move* across the $(n+1)$-ball A, if the interior of A does not meet $|M_1|$, $|M_2|$, and the boundary of A has $\overline{M_1 - M_2}, \overline{M_2 - M_1}$ as an equatorial decomposition. Two n-manifolds M_1, M_r embedded in E^k, $k > n$, are *equivalent by cellular moves* if there is a sequence $M_1, M_2, \cdots, M_r$ of n-manifolds in E^k, each one either differing from the next by a cellular move, or else having the same underlying polyhedron as the next.

In the above definitions we have not required the manifolds to be closed. If a manifold has boundary, then in a simplicial or cellular move the boundary must remain fixed, so that the boundaries of manifolds which are equivalent by either process must have the same underlying polyhedron.

LEMMA 1. *Two n-manifolds embedded in E^k, $k > n$, are equivalent by simplicial moves if and only if they are equivalent by cellular moves.*

The proof one way is trivial, because if two manifolds differ by a simplicial move then *a fortiori* they differ by a cellular move. For the proof the other way we first need a lemma about stellar subdivision.

Subdivision

We use the definitions and notation of Whitehead [6]. All complexes are assumed to be rectilinearly embedded in some euclidean space, and L is *subdivision*[4] of K if every simplex of L is contained in some simplex of K. An *elementary subdivision* of a complex K consists of starring a single simplex $A \in K$ at an interior point a, or in other words replacing $K = AP + Q$ by $K^* = a\dot{A}P + Q$, where P is the link of A in K, and $Q = K - AP$. A *stellar subdivision* of K is the result of a finite sequence of elementary subdivisions. If $L \subset K$, a stellar subdivision of K, which we shall denote by σK, induces a stellar subdivision σL of L.

Consider an embryo simplicial move: let A be an $(n+1)$-simplex (in some E^k, $k > n$) that is the join $A = BC$ of a p-simplex B and a q-simplex C, where $p, q \geqq 0$, $p + q = n$. Then (B, C) is a simplicial move across A between the bounded n-manifolds $B\dot{C}$, $\dot{B}C$.

LEMMA 2. *If σA is a stellar subdivision of A, then $\sigma(B\dot{C})$, $\sigma(\dot{B}C)$ are equivalent by simplicial moves across the simplexes of σA.*

PROOF. The proof is by induction on the number r of elementary subdivisions in σ. First consider the case $r = 1$. Suppose that σA consists

[4] This is called a *partition* in [1] and [5].

of starring the t-dimensional face $X = x_0x_1\cdots x_t$ of A at the interior point x, where $t \geqq 1$. Suppose that the vertices of X are labelled so that the first s of them lies in B, and the remaining $t+1-s$ lie in C, where $0 \leqq s \leqq t+1$. We can write $B = Dx_0x_1\cdots x_{s-1}$, $C = x_sx_{s+1}\cdots x_tE$, where D, E are (possibly empty) simplexes. Then $\sigma(B\dot{C})$ is equivalent to $\sigma(\dot{B}C)$ by the simplicial moves $m_t, m_{t-1}, \cdots, m_0$, in that order, across the $t+1$ $(n+1)$-dimensional simplexes of σA, where m_i is the move

$$m_i = \begin{cases} (Dxx_0x_1\cdots x_{i-1},\ x_{i+1}x_{i+2}\cdots x_tE)\,, & i = 0, 1, \cdots, s-1; \\ (Dx_0x_1\cdots x_{i-1},\ x_{i+1}x_{i+2}\cdots x_txE)\,, & i = s, s+1, \cdots, t. \end{cases}$$

We now prove the inductive step. Suppose that r is the number of elementary subdivisions in σ, so that $\sigma A = \sigma_1\sigma_2 A$, where σ_2 comprises the first $r-1$ elementary subdivisions, and σ_1 is the last elementary subdivision, obtained by starring the t-simplex X in $\sigma_2 A$, say. By induction the lemma holds for $\sigma_2 A$. The required sequence of simplicial moves across σA is obtained from that across $\sigma_2 A$, by replacing each move across an $(n+1)$-simplex in the star of X by $t+1$ moves as in the case $r=1$ above.

Proof of Lemma 1

Suppose M_1, M_2 are two n-manifolds in E^k which differ by a cellular move across the $(n+1)$-ball Q, whose interior does not meet $|M_1 + M_2|$, and whose boundary $\dot{Q}$ has $\overline{M_1 - M_2}$, $\overline{M_2 - M_1}$ as an equatorial decomposition.

Let A be an $(n+1)$-simplex, B a vertex of A, and C the n-dimensional face opposite B. Let $f : A \rightarrow Q$ be a piecewise linear homeomorphism throwing $B\dot{C}$, $\dot{B}C$ $(=C)$ onto $\overline{M_1 - M_2}$, $\overline{M_2 - M_1}$, respectively (one can easily be constructed using [1, Theorem 13.2]). For suitable subdivisions βA, γQ of A, Q the map $f : \beta A \rightarrow \gamma Q$ is a simplicial isomorphism. By [5, Theorem 1], there is a subdivision $\alpha\beta A$ of βA that is a stellar subdivision of A. By Lemma 2, $\alpha\beta(B\dot{C})$, $\alpha\beta(\dot{B}C)$ are equivalent by simplicial moves across the simplexes of $\alpha\beta A$. The simplicial isomorphism $f : \alpha\beta A \rightarrow \alpha\gamma Q$ carries this over to an equivalence, e say, between $\alpha\gamma(\overline{M_1 - M_2})$, $\alpha\gamma(\overline{M_2 - M_1})$ by simplicial moves across the simplexes of $\alpha\gamma Q$. Extend the subdivision $\alpha\gamma Q$ to subdivisions δM_1, δM_2 of M_1, M_2. Since the interior of Q does not meet $|M_1|$, $|M_2|$, the simplicial moves of e can be regarded as simplicial moves between δM_1, δM_2. Therefore M_1, M_2 are equivalent by simplicial moves $M_1 \longrightarrow \delta M_1 \xrightarrow{e} \delta M_2 \longrightarrow M_2$. By induction on the number of cellular moves, if M_1, M_r are equivalent by cellular moves, then they are equivalent by simplicial moves, and the proof of Lemma 1 is complete.

COROLLARY TO LEMMA 1. *If an n-sphere S^n in E^k bounds an $(n+1)$-ball, then it is equivalent by simplicial moves to the boundary of an $(n+1)$-simplex.*

PROOF. Let S^n bound the ball Q. Let A be an n-simplex in S^n, and let $B = Ax$ be the unique $(n+1)$-simplex of Q containing A. We may assume, having first subdivided Q if necessary, that x lies in the interior of Q. Then S^n differs from $\dot{B}$ by a cellular move across $\overline{Q - B}$, which is an $(n+1)$-ball by [1, Corollary 14.5b]. By Lemma 1, S^n is equivalent to $\dot{B}$ by simplicial moves.

LEMMA 3. *If two n-manifolds M_1, M_2 embedded in E^k, $k > n$, differ by a simplicial move across the simplex A, then there is a piecewise linear homeomorphism of E^k onto itself throwing M_1 onto M_2, which is the identity outside an arbitrarily small neighbourhood of A.*

COROLLARY. *If two n-manifolds M_1, M_r embedded in E^k, $k > n$, are equivalent by simplicial moves, then there is a piecewise linear orientation preserving homeomorphism of E^k onto itself, throwing M_1 onto M_r.*

PROOF. The corollary follows from the lemma by composing the piecewise linear homeomorphisms given by the individual simplicial moves.

For the proof of the lemma, suppose that the simplicial move is (B, C) across the $(n+1)$-simplex $A = BC$. Let a, b, c be the barycentres of A, B, C, respectively; if X denotes the 1-simplex bc, then X contains a. Now $X\dot{B}\dot{C}$ is a subdivision of A, and meets $M_1 \cap M_2$ in $\dot{B}\dot{C}$. Since $M_1 \cap M_2$ is a finite complex, there is an arbitrarily small ε-neighbourhood W of X, such that the join of any point in W to $\dot{B}\dot{C}$ also meets $M_1 \cap M_2$ in $\dot{B}\dot{C}$. Let $Y = b'c'$ be a 1-simplex in W, that contains X in its interior, and such that $b'bcc'$ is the order of points in Y. Let Z be a $(k-n-1)$-simplex in W, that is perpendicular to A and has the same barycentre a as A. Let $N = Y\dot{Z}\dot{B}\dot{C}$.

Then N is a convex k-dimensional set in E^k, and is a closed neighbourhood of the interior of A. Since N is contained in the ε-neighbourhood of A, and ε was arbitrarily small, we can choose N to be the arbitrarily small neighbourhood of A mentioned in the statement of the lemma. Since $Y\dot{Z} \subset W$, N meets $M_1 \cap M_2$ in $\dot{B}\dot{C}$. Let $f: Y \to Y$ be the piecewise linear homeomorphism that maps the segments $b'b$, bc' linearly onto the segments $b'c$, cc', respectively. Then $f \mid \dot{Y}$ is the identity. Define f to be the identity on $\dot{Z}\dot{B}\dot{C}$, and extend f linearly to $N = Y\dot{Z}\dot{B}\dot{C}$. Then $f: N \to N$

is a piecewise linear homeomorphism throwing $B\dot{C}$ onto $\dot{B}C$, such that $f|\dot{N}$ is the identity. Define f to be the identity on $E^k - N$. Then f is the required piecewise linear homeomorphism of E^k onto itself, throwing M_1 onto M_2.

THEOREM 1. *Let S^n be a combinatorial n-sphere embedded in euclidean k-space E^k, $k > n$. Then the following five statements are equivalent*:

(1) *S^n in unknotted.*

(2) *S^n bounds a locally unknotted combinatorial $(n + 1)$-ball.*

(3) *S^n bounds a combinatorial $(n + 1)$-ball.*

(4) *S^n is equivalent by cellular moves to the boundary of a combinatorial $(n + 1)$-ball.*

(5) *S^n is equivalent by simplicial moves to the boundary of an $(n + 1)$-simplex.*

PROOF. (1) implies (2) because there is a piecewise linear homeomorphism of E^k onto itself, throwing the boundary of an $(n + 1)$-simplex onto S^n, and throwing the simplex onto a locally unknotted ball bounded by S^n. (2) implies (3) *a fortiori*. (3) implies (4) *a fortiori*. (4) implies (5) by Lemma 1 and its Corollary. (5) implies (1) by the Corollary to Lemma 3. The proof of Theorem 1 is complete.

Example

That (3) implies (2) in Theorem 1 is at first sight mildly surprising; let us try and explain this in an example. Let S^1 be a familiar knotted polygon in $E^3 \subset E^4$, and V a vertex in $E^4 - E^3$. Then S^1 bounds the disk VS^1, which is locally knotted at V. The question is how do we construct a locally unknotted disk D spanning S^1? Artin points out in [2] that the local knottedness of VS^1 is an isotopy invariant, and yet Theorem 1 in effect says that we somehow use VS^1 to construct D. The secret lies in Lemmas 1 and 3.

Let $A_1, A_2, \cdots, A_r$ be the 1-simplexes of S^1, in order; let B_i be the disk $B_i = V(A_1 + A_2 + \cdots + A_i)$, $i = 1, 2, \cdots, r$. We can pass (Lemma 1) from the boundary $\dot{B}_1$ of the triangle B_1 to $\dot{B}_r = S^1$ by the simplicial moves $\dot{B}_1 \to \dot{B}_2 \to \cdots \to \dot{B}_r$. For $1 \leqq i < r$, B_i is in fact a locally unknotted disk spanning $\dot{B}_i$, with V on the boundary; only $B_r = VS^1$ has V interior, and is locally knotted. Lemma 3 meanwhile has been giving us a sequence of locally unknotted disks D_i spanning $\dot{B}_i$, $i = 1, 2, \cdots, r$, images of $D_1 = B_1$ under the piecewise linear homeomorphisms; in general D_i is arbitrarily close to, but not equal to, B_i, $i > 1$. The final piecewise linear homeomorphism of Lemma 3, which throws $\dot{B}_{r-1}$ onto $\dot{B}_r$ and D_{r-1} onto D_r, moves

V, which hitherto has remained fixed, to the barycentre of A_r. Thus D_r has avoided the local knottedness of $B_r = VS^1$ by, as it were, pushing it to the boundary, where it does not matter.

The rest of the paper is devoted to proving:

THEOREM 2. (THE UNKNOTTING THEOREM). *If $k > (3/2)(n+1)$, every combinatorial S^n in E^k is unknotted.*

By Theorem 1 it suffices to show that every S^n it equivalent by cellular moves to the boundary of an $(n+1)$-ball, but before we can do this we need some definitions and lemmas.

Definitions

Let the *hull* of a q-simplex in E^k be the linear q-dimensional subspace which it spans. Let the *hull of a pair* of simplexes be the linear subspace spanned by their two hulls; if this coincides with the whole of E^k we say the simplexes are *skew*. Notice that if two simplexes are skew, the sum of their dimensions must be $\geqq k-1$. Given three simplexes, define a *proper transversal* of them to be a line which meets each simplex and its hull in exactly one point, and such that the three points are distinct. Define the *transversal set* of the three simplexes to be the union of all proper transversals. Notice that the transversal set as we have defined it is a fairly awkward kind of set, possibly empty, and probably neither open nor closed. However it is contained in the hull of each pair; and although it is not in general linear, it is contained in an algebraic variety, so that we can deduce:

LEMMA 4.[5] *If three simplexes of dimension $\leqq n$ lie in E^k, $k \geqq (3/2)(n+1)$, then their transversal set is contained in an algebraic variety of dimension $< k$.*

PROOF. We may assume all three simplexes to be of dimension n, because if we prove the lemma for this case, it follows for any lesser dimensions. Also we may assume the three simplexes to be pairwise skew, otherwise the transversal set is contained in the hull of a non-skew pair, which is of dimension $< k$. Let A, B, C be the three simplexes, and X, Y, Z their hulls. Suppose that a proper transversal meets A in x. If $[x, Y]$ denotes the linear subspace spanned by x and Y, then $\dim [x, Y] = n+1$. Since Y, Z together span E^k, $\dim ([x, Y] \cap Z) = 2n+1-k$. Hence $\dim [x, ([x, Y] \cap Z)] = 2n+2-k$. But $[x, ([x, Y] \cap Z)]$ contains all the proper transversals through x. Therefore

[5] Whereas Lemma 4 holds for $k \geqq (3/2)(n+1)$, our proof of Theorem 2 only holds for $k > (3/2)(n+1)$. See the Remark after Lemma 5.

the transversal set of the three simplexes is contained in the algebraic variety traced out by the subspaces $[x, ([x, Y] \cap Z)]$ as x varies through X, which is of dimension $\leqq n + (2n + 2 - k) \leqq k - 1$.

Beginning of the proof of Theorem 2

We are given a combinatorial n-sphere S^n embedded in E^k, $k > (3/2)(n + 1)$, which we have to show is unknotted. Let V be a vertex in general position. The word "general" merely implies that V does not lie in any of the following finite set of subspaces of E^k, each of which has dimension less than k ((iii) by Lemma 4):

(i) The hull of any simplex in S^n.

(ii) The hull of any non-skew pair of simplexes in S^n.

(iii) The transversal set of any three simplexes in S^n.

Also since $|S^n|$ is compact, we can arrange for V to have a further property, which will be technically convenient:

(iv) V lies on some $(k - 1)$-dimensional linear subspace which does not meet S^n.

Singular points

Call a point $x \in |S^n|$ *non-singular* if Vx does not meet $|S^n|$ again; otherwise call it *singular*. Let Ω be the set of singular points. If Ω is empty we are finished by Theorem 1, because the cone VS^n is then a combinatorial $(n + 1)$-ball bounded by S^n. Therefore assume Ω is non-empty; we shall eventually make a cellular move of S^n in order to remove the singularities $V\Omega$ of the cone.

Suppose x, y are singular points collinear with V. The general position of V ensures that they can not lie in the same simplex (by (i) above); that they lie in skew simplexes (by (ii)); and that they are not collinear with any other singular point (by (iii)). Also property (iv) ensures that they both lie on the same side of V, and so we can call the one nearer to V a *near-singular* point, and the other a *far-singular* point. Let Ω_1 be the set of all near-singular points, and Ω_2 the set of all far-singular points. Then $\Omega = \Omega_1 \cup \Omega_2$ the union of two disjoint subsets. Notice that the limit of singular points is not necessarily singular, and so in general Ω_1, Ω_2 are neither open nor closed in $|S^n|$.

LEMMA 5. *We can separate Ω_1 and Ω_2 by an equator. More precisely, there is a subdivision αS^n of S^n, and an equatorial decomposition $B_1 + B_2 = \alpha S^n$, $B_1 \cap B_2 = S^{n-1}$ of αS^n, such that Ω_1 lies in the interior of B_1 and Ω_2 lies in the interior of B_2.*

REMARK. The reason that we shall be able to prove Lemma 5 is that Ω is of sufficiently low dimension, due to the restriction $k > (3/2)(n+1)$. In fact Lemma 6, which we use to prove Lemma 5, dictates the lower bound for k in the statement of Theorem 2. If $k = (3/2)(n+1)$ we can still form the sets Ω_1 and Ω_2 because Lemma 4 holds, but they may link, and so Lemma 5 fails. Possibly this means that S^n can be knotted in E^k when $k = (3/2)(n+1)$. The proof of Lemma 5 is easy when $k = 2n+1$ (for instance S^2 in E^5) because Ω turns out to be just a finite set of points. In the marginal case, however, the proof is tricky because the closures of Ω_1 and Ω_2 may intersect (in non-singular points), and so S^{n-1} must go through this intersection. For example if $n = 2m$, $k = 3m+2$, then this intersection may contain much of the $(m-2)$-skeleton of S^n.

Proof of Lemma 5

Let A, B be two skew closed simplexes of S^n (i.e., their hull is E^k). Then $\dim A + \dim B \geqq k-1$, and so $\dim A \geqq k-n-1$. Also $\dim (A \cap B) \leqq 2n-k < k-n-3$. Therefore if $A \cap B$ is non-empty; it is a simplex which cannot be skew to any other simplex of S^n, and so contains no singular points (by (ii) above).

Let the transversal set of V, A, B meet A, B in X, Y, respectively. If X, Y are non-empty they are homeomorphic linear convex sets of dimension $\leqq 2n-k+1 < \frac{1}{2}(n-1)$, since $k > (3/2)(n+1)$. However X, Y may not be closed, because we restricted our definition of transversal set to be the union of only proper transversals, which have to meet A, B in distinct points. To obtain the closures of X, Y we may have to add the intersections of some transversals which meet $A \cap B$. Therefore

$$\bar{X} - X = \bar{Y} - Y = \bar{X} \cap \bar{Y} \subset A \cap B \subset |S^n| - \Omega .$$

The three sets $\bar{X}, \bar{Y}, \bar{X} \cap \bar{Y}$ are triangulable, and have dimension $< \frac{1}{2}(n-1)$. We choose the order of the pair A, B so that $X \subset \Omega_1$, $Y \subset \Omega_2$. Now take the union over all such ordered skew pairs A, B. We obtain three triangulable set of dimension $< \frac{1}{2}(n-1)$:

$$\bigcup \bar{X} = \bar{\Omega}_1$$

$$\bigcup \bar{Y} = \bar{\Omega}_2$$

$$\bigcup (\bar{X} \cap \bar{Y}) = \bar{\Omega}_1 \cap \bar{\Omega}_2 = \bar{\Omega}_1 - \Omega_1 = \bar{\Omega}_2 - \Omega_2 \subset |S^n| - \Omega .$$

Let L be a subdivision of S^n containing subcomplexes $L_1, L_2, L_3 = L_1 \cap L_2$ of dimension $< \frac{1}{2}(n-1)$ triangulating these three sets $\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_1 \cap \bar{\Omega}_2$, respectively.

We now transfer the situation from k dimensions to n dimensions. Let

$f: L \to \Delta^{n+1}$ be a piecewise linear homeomorphism throwing $L_1 + L_2$ into the interior of an n-dimensional face Δ^n of Δ^{n+1} (this being possible since $L_1 + L_2 \neq L$). Let $K_i = fL_i$, $i = 1, 2, 3$. We now appeal to Lemma 6 below. Let αS^n be a subdivision of S^n that contains a subdivision B_1 of $f^{-1} B$ as a subcomplex, and let $B_2 = \overline{\alpha S^n - B_1}$. Then, by [1, Theorem 14.2], B_1, B_2 is an equatorial decomposition of αS^n, and by construction the interiors of B_1, B_2 contain Ω_1, Ω_2, respectively. The proof Lemma 5 is complete, subject to:

LEMMA 6. *Let K_1, K_2 be two finite complexes of dimension $< \frac{1}{2}(n-1)$ in the interior of Δ^n, and let $K_3 = K_1 \cap K_2$. Then there is a combinatorial n-ball B in Δ^n, whose interior contains $|K_1| - |K_3|$, whose boundary contains K_3, and whose exterior contains $|K_2| - |K_3|$.*

PROOF.[6] Let Q be a vertex in general position in Δ^n, Q does not lie in the hull of any pair of simplexes in $K_1 + K_2$, because any such hull is of dimension less than n. Therefore no two distinct points of $|K_1 + K_2|$ are collinear with Q. Hence the join QK_1 is a non-singular cone on K_1, and $QK_1 \cap K_2 = K_1 \cap K_2 = K_3$. We elongate the cone QK_1 to a simplicially isomorphic cone QK in the interior of Δ^n, keeping the subcone QK_3 fixed, as follows: to each vertex $x_1 \in K_1$ define a vertex $x \in K$, such that $x = x_1$ if $x_1 \in K_3$ and Qx_1x are collinear with $x \neq x_1$ if $x_1 \in K_1 - K_3$. Join up the vertices x with simplexes in 1–1 correspondence with those of K_1, to form the complex K. The purpose of this is to get $|K_1| - |K_3| \subset |QK| - |K|$. Meanwhile if $x \in |K|$ and $y \in |K_2| - |K_3|$, then $y \notin Qx$, and so $QK \cap K_2 = K_3$.

Let $\beta\Delta^n$ be a subdivision of Δ^n containing QK, K_2, and hence K_3, as subcomplexes. Let $M = s_{QK}{}^2(\beta\Delta^n)$, the second derived complex of $\beta\Delta^n$ modulo QK, in the sense of [6, page 251]. Let $B = N(QK - K, M)$, the union of the closed stars in M of all simplexes in $QK - K$. Any point of $|K_1| - |K_3|$ lies in the interior of some simplex of $QK - K$, and hence in the interior of its star. Therefore $|K_1| - |K_3|$ lies in the interior of B. If $A \in K_1 - K_3$, $C \in K_2 - K_3$, then any simplex of $\beta\Delta^n$ containing C is subdivided twice in M, and so the closed star of A in M cannot contain any interior points of C. Therefore C, and consequently $|K_2| - |K_3|$, lie in the exterior of B. It follows that K_3 lies in the boundary of B. To complete the proof of Lemma 6, it remains to verify that B is a ball.

We use the techniques of [6, § 12] to prove by induction on n that B is an n-ball. The inductive hypothesis is that if QK is a cone in an n-manifold M, satisfying conditions (by [6, Lemma 4])

(i) no simplex in $M - QK$ has all its vertices in QK,

[6] I am indebted to Henry Whitehead for simplifying this proof.

(ii) if $A \in M - QK$ then $QK \cap \mathrm{lk}(A, M)$ is either empty or a closed simplex, then $B = N(QK - K, M)$ is an n-ball.

The induction starts trivially when $n = 0$. Assume the statement true for $n-1$. Then it suffices to prove that B is an n-manifold, because by (i) and (ii) and [6, Theorem 2], B is then a regular neighbourhood of QK, which is collapsible by [6, Corollary to Lemma 2], and so B is an n-ball by [6, Theorem 23 Corollary 1].

Therefore we must verify that if x is a vertex of B, then $\mathrm{lk}\,(x, B)$ is either an $(n-1)$-sphere or an $(n-1)$-ball. Let $M^* = \mathrm{lk}\,(x, M)$; then certainly M^* is either an $(n-1)$-sphere or an $(n-1)$-ball, because M is manifold. If $x = Q$ then $\mathrm{lk}\,(x, B) = M^*$. If $x \in K$, then $\mathrm{lk}\,(x, B) = N(QK^* - K^*, M^*)$, where $K^* = \mathrm{lk}\,(x, K)$; the pair M^*, QK^* satisfy conditions (i) and (ii) by inheritance from M, QK, and so by induction $\mathrm{lk}\,(x, B)$ is an $(n-1)$-ball. Finally if $x \in B - QK$, then $\mathrm{lk}\,(x, B) = \overline{\mathrm{st}\,(Q, M^*)}$, which is an $(n-1)$-ball. The proof of Lemma 6 is complete.

Completion of the proof of Theorem 2

Using Lemma 5, define $T^n = VS^{n-1} + B_2$. Then T^n is a sphere, because S^{n-1} is comprised of only non-singular points and the cone VS^{n-1} is an n-ball meeting the n-ball B_2 only in their common boundary S^{n-1}. The cone VB_1 is an $(n+1)$-ball, because B_1 contains only non-singular and near-singular points. The move from αS^n to T^n is a cellular move across this ball, because the boundary of VB_1 has an equatorial decomposition $B_1 = \overline{\alpha S^n - T^n}$, $VS^{n-1} = \overline{T^n - \alpha S^n}$, and the interior of VB_1 does not meet the spheres $|S^n|, |T^n|$, because the only points where it might do so are the far-singular points, but these are too far away. Meanwhile T^n bounds the non-singular $(n+1)$-ball VB_2. Therefore S^n is equivalent by cellular moves $S^n \to \alpha S^n \to T^n$ to the boundary of a ball, and so is unknotted by Theorem 1.

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