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IMBEDDING OF MANIFOLDS IN EUCLIDEAN SPACE

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1. The main theorems

It is obvious, since the vertices may be placed in general position, that a finite, n-dimensional simplicial complex can be piecewise linearly imbedded in euclidean $(2n+1)$-space, R^{2n+1}. This is the best possible result for an arbitrary complex since the n-skeleton of a $(2n+2)$-simplex cannot be imbedded in R^{2n} [4], [9]. On the other hand a compact, smooth or combinatorial n-manifold (see § 2 for definitions) can be (smoothly or piecewise linearly) imbedded in R^{2n} [13], [19], [20]. Real projective n-space cannot be smoothly imbedded in R^{2n-1} if $n = 2^k$ [14], [21], though there are better results for certain other projective spaces [8].

In this paper we are concerned with piecewise linear imbeddings in R^q of compact, n-dimensional, combinatorial manifolds (see § 2) which are $(m-1)$-connected, where $0 < 2m \leqq n$. The condition $m > 0$ means that such a manifold is connected. If a closed (i.e., compact, unbounded) n-manifold M is $(m-1)$-connected and $2m > n$, then it follows from the Poincaré duality that M has the homotopy type of an n-sphere. Therefore, if it turns out that every such manifold is a (combinatorial) n-sphere, or even if it can be piecewise linearly imbedded in R^{n+1}, then (1.1) below is valid for $0 < m \leqq n$. Except when the contrary is stated, it is to be understood that all the manifolds to which we refer are combinatorial and that all our maps, in particular the immersions (see § 2) and imbeddings, are piecewise linear. We prove:

THEOREM (1.1). *If $0 < 2m \leqq n$, then every closed, $(m-1)$-connected n-manifold can be imbedded in R^{2n-m+1}.*

THEOREM (1.2). *Let M be a compact, bounded n-manifold which is $(m-1)$-connected $(0 < 2m \leqq n)$. If either*

(a) *$\dot{M} \times I$ can be imbedded in R^{2n-m}, or*

(b) *$\dot{M}$ is $(m-2)$-connected ((-1)-connected means non-vacuous), then M can be imbedded in R^{2n-m}.*

It follows from (2.3) that the condition (a) is necessary for the imbeddability of M in R^{2n-m}. It is obviously satisfied if $\dot{M}$ can be imbedded in R^{2n-m-1}, hence, by (1.1), if each component of $\dot{M}$ is $(m-1)$-connected and $2m < n$.

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By the *branch locus* of a map $f: M \to R^q$ we mean the set of points $x \in M$ such that no neighbourhood of x is imbedded by f. Let K be a (rectilinear) triangulation of M such that f is barycentric in each simplex of K (we do not assume that f is simplicial with respect to K and a triangulation of R^q). Then the branch locus, B, of f is the union of all the closed simplexes $\sigma \in K$ such that $f | \mathrm{St}(\sigma, K)$ is not an imbedding, where $\mathrm{St}(\sigma, K)$ denotes the union of all the closed simplexes of K which contain σ. Therefore B is a compact polyhedron. Clearly $f | M - B$ is an immersion. We shall prove:

THEOREM (1.3). *Let M be a closed, $(m-1)$-connected n-manifold, where $0 < 2m \leqq n$. Assume that there is a map $M \to R^q$ whose branch locus is at most $(m-1)$-dimensional, where $q > 2(n-m)$. Then M can be imbedded in R^{q+1}.*

In particular M can be imbedded in R^{q+1} if it can be immersed in R^q, provided M satisfies the conditions of (1.3) and $q > 2(n-m)$. Thus if $n = 2m$ and M can be immersed in R^q (whence $q > n$ because M is closed), then it can be imbedded in R^{q+1}.

PROOF OF (1.1), *assuming* (1.3). Let $f: M \to R^{2n-m}$ be a map which is barycentric in each simplex of a triangulation K of M and which maps the vertices of K in general position. Then f imbeds each simplex of K. Let σ_1, σ_2 be simplexes of K, let $\sigma_0 = \sigma_1 \cap \sigma_2$ and let $\dim \sigma_0 = p \geqq m$, $\dim \sigma_1 = r$, $\dim \sigma_2 = s$. Then σ_1 is the join of σ_0 and an $(r-p-1)$-simplex τ. Since

$$(r-p-1) + s - (2n-m) \leqq m - p - 1 < 0$$

and the vertices of K are mapped in general position, it follows that f does not meet the s-plane containing $f\sigma_2$. Therefore $f\sigma_1 \cap f\sigma_2 = f\sigma_0$ and it follows that $f | \mathrm{St}(\sigma, K)$ is an imbedding if $\dim \sigma \geqq m$ $(\sigma \in K)$. Therefore the branch locus of f is at most (m $-$ 1)-dimensional and (1.1) follows from (1.3) since $2n - m > 2(n-m)$.

Let P, Q be compact polyhedra in a manifold M. We describe Q as *quasi-complementary* to P if, and only if, every compact polyhedron in $M - P$ can be (piecewise linearly) imbedded in every neighbourhood of Q. For example, let K be a triangulation of M and let the vertices of K be separated into two disjoint subsets A, B. Let P be the union of the simplexes of K whose vertices are all in A and Q the union of the simplexes of K whose vertices are all in B. Then P, Q are quasi-complementary to each other [2]. In particular, if L is the cell-complex dual to K, then it follows from the preceding remark, applied to the first barycentric subdivision of K, that $|L^{n-m}|$ is quasi-complementary to $|K^{m-1}|$

(X^r denotes the r-skeleton of a given complex X and $|X|$ denotes the polyhedron covered by a given complex X). In general P and Q may have points in common. For example, if M is a (combinatorial) n-sphere, then P, Q may be any proper, non-vacuous, compact, polyhedral subsets of M.

THEOREM (1.4). *Let M be a compact (possibly bounded), $(m-1)$-connected, n-manifold, where $0 < 2m \leqq n$. Assume that there are compact polyhedra $P \subset \operatorname{int} M$, $Q \subset M$ such that $\dim P < m$, Q is quasi-complementary to P and some neighbourhood of Q can be imbedded in R^q. Then M can be imbedded in R^{q+1} and in R^q if it is bounded.*

PROOF. Let $U \subset M$ be a neighbourhood of Q which can be imbedded in R^q. Since $P \subset \operatorname{int} M$, $\dim P < m$ and M is $(m-1)$-connected it follows from (2.9) below that, if M is bounded, it can be imbedded in $M - P$, hence in U and hence in R^q.

Let M be unbounded. Then, by (2.7), there is an n-element $E \subset M$ such that $P \subset \operatorname{int} E$. Let $M_0 = M - \operatorname{int} E$. Then M_0 can be imbedded in U and there is therefore an imbedding $f: M_0 \to R^q$. We take R^q to be a hyperplane in R^{q+1} and extend f to an imbedding $M \to R^{q+1}$, which maps E on the join of $f\dot{E}$ and a point in $R^{q+1} - R^q$. This proves (1.4).

LEMMA (1.5). *Let Q be a compact polyhedron in a manifold M and let Q have a neighbourhood which can be immersed in R^q, where $q > 2 \dim Q$. Then Q has a neighbourhood which can be imbedded in R^q.*

This follows from the properties of general position and from (2.1).

PROOF OF (1.3). Let $f: M \to R^q$ be a map whose branch locus, B, is at most $(m-1)$-dimensional and let K be a triangulation of M such that f is barycentric in each simplex of K. Let L be the cell-complex on M which is dual to K. Then $f \mid M - B$ is an immersion, $B \subset K^{m-1}$ and (1.3) follows from (1.5), (1.4) with $P = |K^{m-1}|$, $Q = |L^{n-m}|$.

Let M be a homotopy n-sphere (i.e., a combinatorial manifold of the homotopy type of an n-sphere), let E be an n-element in M and let $M_0 = M - \operatorname{int} E$. Then it follows from a theorem in a forthcoming paper by A. M. Gleason that there is an immersion $f: M_0 \to R^n$. Let $h: E \to \Delta$ be a homeomorphism of E on an n-simplex Δ, let $a \in \operatorname{int} \Delta$, $b \in R^{n+1} - R^n$ and let $k: \Delta \to R^{n+1}$ be defined by

$$k((1-t)a + thx) = (1-t)b + tfx \qquad (x \in \dot{E},\ t \in I)\ .$$

Then a map $g: M \to R^{n+1}$ is defined by $gx = fx$ or khx according as $x \in M_0$ or $x \in E$. Since f is an immersion it follows that the branch locus of g consists, at most, of the point $h^{-1}a$. Therefore we have, by (1.3):

THEOREM (1.6). *A homotopy n-sphere can be imbedded in R^{n+2} if $n = 2m$ and in R^{n+3} if $n = 2m+1$.*

We describe M as *combinatorially equivalent* to a smooth manifold M_1 if, and only if, some triangulation, K, of M is the argument complex in a C^1-triangulation of M_1 [17]. This means that there is a homeomorphism $f: M \to M_1$ such that, for every closed n-simplex $\sigma \in K$, the map $f \mid \sigma$ can be extended to a regular C^1-imbedding $U(\sigma) \to M_1$, where $U(\sigma)$ is an open neighbourhood of σ in the n-plane which contains it. The manifold M_1 is said to be *almost parallelizable* if, and only if, $M_1 - p$ is parallelizable for some, and therefore every point $p \in M_1$. Let M be closed and let K and $f: M \to M_1$ be as above, where M_1 is almost parallelizable. Let M be $(m-1)$-connected $(0 < 2m \leqq n)$, let $E \subset M$ be an n-element such that $K^{m-1} \subset \operatorname{int} E$ and let $M_0 = M - \operatorname{int} E$. Then fM_0 is parallelizable. Therefore there is a smooth immersion $fM_0 \to R^{n+1}$ [6, p. 269]. Let K_0 be a triangulation of M_0 such that, for every simplex $\sigma \in K_0$, the restriction of $fM_0 \to R^{n+1}$ to some neighbourhood of $f\operatorname{St}(\sigma, K_0)$ is a regular imbedding. Then it follows from Theorems 1 and 3 in [17] (Theorem 3 may be applied to every $\operatorname{St}(\sigma, K_0)$) that M_0 can be piecewise linearly immersed in R^{n+1} and hence in $R^{2(n-m)+1}$. Let L be the cell-complex dual to K and let $Q = |L^{n-m}|$. Then, as in the proof of (1.3), M_0 can be imbedded in every neighbourhood of Q and some neighbourhood of Q can be imbedded in $R^{2(n-m)+1}$. Therefore we have, by an argument in the proof of (1.4):

THEOREM (1.7). *Let M be a closed, $(m-1)$-connected n-manifold $(0 < 2m \leqq n)$, which is combinatorially equivalent to an almost parallelizable smooth manifold. Then M can be piecewise linearly imbedded in $R^{2(n-m)+2}$.*

Theorems (1.6), (1.7) were pointed out to us by M. W. Hirsch.

PROOF OF (1.2). Let $\dot{M} \times I$ be imbeddable in R^{2n-m}. By (2.3) there is a closed, polyhedral neighbourhood $N \subset M$ of $\dot{M}$ which is (piecewise linearly) homeomorphic to $\dot{M} \times I$ and hence imbeddable in R^{2n-m}. Let K be a triangulation of the pair (M, N) (i.e., a triangulation of M with a subcomplex covering N) such that a given imbedding $N \to R^{2n-m}$ is barycentric in each simplex of $K \cap N$. Extend this to a map $f: K \to R^{2n-m}$ which is barycentric in each simplex of K. We assume, as we obviously may, that $\dot{K}$ is a *full subcomplex* of K (i.e., that every simplex of K with all its vertices in $\dot{K}$ is contained in $\dot{K}$) and that f maps the vertices of K in general position. Let P denote the union of all the closed simplexes of K which are at most $(m-1)$-dimensional and do not meet $\dot{M}$. Then the branch locus of f is contained in P.

Let K' denote the first barycentric subdivision of K. The simplexes of K' are of the form $c(\sigma_0)\cdots c(\sigma_p)$, where $c(\sigma)$ denotes the centroid of a given simplex $\sigma \in K$ and $\sigma_i \subset \sigma_{i+1}$ $(i = 0, \cdots, p-1)$. Let $K_0 \subset K'$ be the subcomplex which consists of all simplexes $c(\sigma_0)\cdots c(\sigma_p) \in K'$ such that $\sigma_0 \not\subset \dot{K}$. Then $K_0 = K' - \mathrm{O}(\dot{K}', K')$, where $\mathrm{O}(\dot{K}', K')$ denotes the union of the open simplexes of K' whose closures meet $\dot{K}'$. Let $M_0 = |K_0|$. Since $\dot{K}$ is a full subcomplex of K it follows from (2.3), applied to both M and M_0, that M_0 is homeomorphic to M. Therefore it is enough to prove that M_0 can be imbedded in R^{2n-m}.

Let us describe a vertex $c(\sigma) \in K_0$ as of the *first kind* if $\dim(\sigma) < m$ and $\sigma \cap \dot{K} = \emptyset$ (σ denotes a closed simplex) and of the *second kind* if either $\dim(\sigma) \geqq m$ or $\sigma \cap \dot{K} \neq \emptyset$. Then the polyhedron P is the union of the simplexes of K_0 whose vertices are all of the first kind. Let Q denote the union of the simplexes of K_0 whose vertices are all of the second kind. Then Q is quasi-complementary to P. It consists of all simplexes $c(\sigma_0)\cdots c(\sigma_p) \in K_0$ such that either $\dim(\sigma_0) \geqq m$, whence $p \leqq n - m$, or $\sigma_0 \cap \dot{K} \neq \emptyset$, which means that $c(\sigma_0)\cdots c(\sigma_p) \in \dot{K}_0$. Moreover $Q \subset M_0 - P$ and $\dot{M}_0 \subset N$, whence $f \,|\, \dot{M}_0$ is an imbedding.

Let the images of the vertices of K_0 be shifted slightly so as to define a map $f_0\colon M_0 \to R^{2n-m}$, barycentric in each simplex of K_0, such that $f_0 | \dot{M}_0$ is an imbedding, $f_0 | M - P$ is an immersion and f_0 maps the vertices of K_0 in general position. Since $\dim(Q - \dot{M}_0) \leqq n-m$ and $(n-m)+(n-1) < 2n - m$ it follows the $f_0 | Q$ is an imbedding. By (2.1) f_0 imbeds some neighbourhood of Q and (1.2) with Hypothesis (a) follows from (1.4).

Now let $\dot{M}$ be $(m-2)$-connected and let M_1 be a copy of M such that $\dot{M}_1 = \dot{M} = M \cap M_1$. Let $M_2 = M \cup M_1$. Then it follows, trivially if $m = 1$ (since $\dot{M} \neq \emptyset$) and from a theorem due to van Kampen [9, p. 177] and from the Mayer-Vietoris theorem [3] if $m > 1$, that M_2 is $(m-1)$-connected. Let E be an n-element in M_1 and let $M_3 = M_2 - \operatorname{int} E$. Then M_3 is $(m-1)$-connected and $\dot{M}_3$ is the $(n-1)$-sphere $\dot{E}$. Therefore it follows from (1.2) with Hypothesis (a) that M_3 can be imbedded in R^{2n-m}. Since $M \subset M_3$ this completes the proof.

The problem of adapting these methods, for closed manifolds, to the smooth theory leads to the following question. Let D^n be the n-disc bounded by a unit $(n-1)$-sphere $S^{n-1} \subset R^n$ and let $f\colon S^{n-1} \to R^q$ be a regular imbedding of class C^r $(1 \leqq r < \infty)$. Can f be extended to a regular C^r imbedding $g\colon D^n \to R^{q+1}$ such that $g(\operatorname{int} D^n) \subset R^{q+1} - R^q$? The answer is "yes" if $q \geqq 2n$, but there is an unpublished theorem due to R. H. Fox and J. W. Milnor (see [5]) which implies that, if $n = 2$, $q = 3$, there are imbeddings f for which the answer is "no". This is because fS^{n-1} is knotted in a certain way and does not bound any regular C^r disc whose interior lies in $R^{q+1} - R^q$. For larger values of n and $q < 2n$ there may, possibly, be cases in which the answer is "no" even though fS^{n-1} is "good", say $fS^{n-1} = S^{n-1}$ $(R^n \subset R^q)$, because f is "bad". (Cf. [10].) For the case of imbeddings of bounded manifolds these difficulties are not always so serious. Some results along these lines have recently been obtained by M. W. Hirsch [7].

Many of the results given here lead to imbeddings which are *locally unknotted*, however [22]. (The piecewise linear imbedding $f\colon M \to R^q$ is locally unknotted if, for triangulations K of fM and L of R^q with K a subcomplex of L, we have $\dot{\mathrm{S}}\mathrm{t}\,(v, K)$ unknotted in $\dot{\mathrm{S}}\mathrm{t}\,(v, L)$ for every vertex $v \in K$. That is, $\dot{\mathrm{S}}\mathrm{t}\,(v, K) = \dot{E}$ for some n-element $E \subset \dot{\mathrm{S}}\mathrm{t}\,(v, L)$.) It follows from the results of [22] that any (piecewise linear) imbedding of an n-manifold M in R^q must be locally unknotted if $2(q-1) > 3n$. Thus, in particular, the imbedding of (1.1) is necessarily locally unknotted unless $2m = n$. Also, it is not hard to adapt the proof of (1.2) to obtain locally unknotted imbeddings in all its cases (although the case $m = 1$, $n = 3$ appears to need special consideration).

2. Definitions and lemmas

A map $f\colon X \to Y$, where X, Y are arbitrary topological spaces, is called an *imbedding* if, and only if, it is a homeomorphism onto fX. A map $f\colon X \to Y$ is called an *immersion* if, and only if, every point $x \in X$ has a neighbourhood $N_x \subset x$ such that $f \mid N_x$ is an imbedding.

LEMMA (2.1). *Let $f\colon X \to Y$ be an immersion of a locally compact metric space X in a Hausdorff space Y and let $f \mid A$ be an imbedding, where A is a compact subset of X. Then there is a compact neighbourhood $N \subset X$ of A such that $f \mid N$ is an imbedding.*

PROOF. Let $N_i = \{x \in X \mid \delta(x, A) \leqq 1/i\}$ $(i = 1, 2, \cdots)$, where δ is a metric for X. Then there is an integer k such that N_i is compact if $i \geqq k$. Assume that $f \mid N_i$ is not an imbedding for any i. Then, there are points $x_i, x_i' \in N_i$ such that $x_i \neq x_i'$, $fx_i = fx_i' = y_i$, say, because N_i is compact if $i \geqq k$ and Y is a Hausdorff space. Since f is locally 1-1 and $f \mid A$ is 1-1 it follows without difficulty that some subsequence of the sequence $\{y_i\}$ converges to each of two distinct points in fA. This is absurd and (2.1) follows.

By a (compact) polyhedron we mean a subspace of R^q, for some q, which can be triangulated by a finite, rectilinear, simplicial complex. It is to be understood that all the triangulations of polyhedra and subdivisions of complexes to which we refer are rectilinear. A map $P \to Q$, where P, Q are polyhedra, is called *piecewise linear* if, and only if, it is simplical with respect to suitable triangulations of P, Q. Thus P, Q are piecewise linearly homeomorphic if, and only if, they have isomorphic triangulations.

This is an equivalence relation because two triangulations of the same polyhedron have a common subdivision. More generally, if P_0, P are polyhedra such that $P_0 \subset P$ and if K_0, K are triangulations of P_0, P, then there is a subdivision of K with a subcomplex which is a subdivision of K_0 [15]. As stated in § 1, all the maps between polyhedra to which we refer will be piecewise linear. Thus "homeomorphic", with reference to polyhedra, will always mean "piecewise linearly homeomorphic". If K is a triangulation of a polyhedron P and if X denotes either a subset of P or a subcomplex of K, then $N(X, K)$ will denote the union of all the closed simplexes of K which meet X. The symbol $N(X, K)$ will denote either a polyhedron or a complex, according to the context (or the choice of the reader).

By an *n-element* (*n-sphere*) we mean a polyhedron which is homeomorphic to a closed n-simplex (boundary of an $(n+1)$-simplex). By a (*combinatorial*) *n-manifold* we mean a polyhedron, M, with a triangulation K such that $\mathrm{St}\,(v, K)$ is an n-element, for every vertex $v \in K$. This property is independent of the choice of K. We denote the boundary of a manifold M by $\dot{M}$ and $\mathrm{int}\, M = M - \dot{M}$.

Let M be a bounded n-manifold and let $E^n \subset M$ be an n-element such that $\dot{M} \cap E^n = E^{n-1}$, say, is an $(n-1)$-element in $\dot{E}^n$. Then $\dot{E}^n - \mathrm{int}\, E^{n-1}$ is also an $(n-1)$-element [1, Theorem 14.2]. Let $M_0 = M - \mathrm{int}\, E^n - \mathrm{int}\, E^{n-1}$. Then we have [11, Theorem 8a], [1, Theorem (14.3)]:

LEMMA (2.2). *M_0 is homeomorphic to M.*

Let M be as in (2.2) and let K be a triangulation of M such that $\dot{K}$ is a full subcomplex of K (see the proof of (1.2)) and let K' denote the first barycentric subdivision of K. Then we have [16]:

THEOREM (2.3). *$N(\dot{M}, K')$ is homeomorphic to $\dot{M} \times I$.*

Let A, B be polyhedra such that $B = A \cup E$, where E is a k-element $(k > 0)$ and $A \cap E$ is a $(k-1)$-element in $\dot{E}$. Then the ordered pair (A, B) will be called an *elementary expansion* (of order k) and (B, A) an *elementary contraction* (of order k). A polyhedron P will be said to *expand* into Q, and Q to *collapse* into P, if, and only if, either $P = Q$ or there is a sequence of elementary expansions (A_i, A_{i+1}) $(i = 1, \cdots, r-1)$, of arbitrary orders, such that $A_1 = P$, $A_r = Q$. A polyhedron P will be called *completely collapsible* if, and only if, it collapses to a point. Obviously an element is completely collapsible. Let P be a polyhedron in an n-manifold M. By a *regular enlargement* (in M) of P we mean an n-manifold, N, such that $P \subset N \subset M$ and N collapses into P. By a *regular neighbourhood* of P we mean a regular enlargement $N \subset M$ of P which is a closed neighbourhood of P (i.e., $P \cap \overline{M - N} = \varnothing$). If P is an n-manifold it is a regular enlargement of itself. Therefore (2.2) is a special case of (2.5) below (the proof of (2.5) depends on (2.2)).

Let K be a triangulation of the pair (M, P) and let K'' be its second barycentric subdivision. Then we have:

THEOREM (2.4). *$N(P, K'')$ is a regular neighbourhood of P.*

THEOREM (2.5). *Any two regular enlargements in M of the same polyhedron are (piecewise linearly) homeomorphic.*

For the proofs of (2.4), (2.5) see [16, p. 293]. We have altered some of the terms used in [16] so as to emphasize the distinction between a "collapse" and an arbitrary retraction by deformation; also to retain the ordinary meaning of "neighbourhood".

If N is a regular enlargement of P and P collapses into P_0, then N is a regular enlargement of P_0. Therefore it follows from (2.5) that every regular enlargement (in M) of P is homeomorphic to every regular enlargement of P_0. By (2.4) a regular enlargement of a point is an n-element. So we have:

COROLLARY (2.6). *Every regular enlargement of a completely collapsible polyhedron is an n-element.*

We now come to the main lemma.

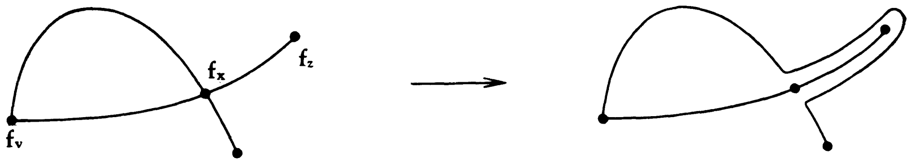
LEMMA (2.7). *Let M be an n-manifold and let $P \subset \operatorname{int} M$ be an $(m-1)$-dimensional polyhedron $(0 < 2m \leqq n)$ such that the inclusion map $i: P \to M$ is homotopic in M to a constant. Then there is an n-element $E \subset \operatorname{int} M$ such that $P \subset \operatorname{int} E$.*

PROOF. Let $C = v * P$ be a cone with P as base $(P \subset C)$ and vertex v. Assume that i can be extended to an imbedding $h: C \to \operatorname{int} M$ and let K be a triangulation of the pair (M, hC). Obviously C, hC are completely collapsible and (2.7), with $E = N(hC, K'')$, follows from (2.4), (2.6). We proceed to prove the existence of h.

Since $i \cong$ const. it can be extended to a (piecewise linear) map $f: C \to \operatorname{int} M$. Let P_0 be a triangulation of P and let C_0 be the triangulation of C which consist of the simplexes $v * \sigma$ and their faces, for every simplex $\sigma \in P_0$. We describe f as *normal* if, and only if, it is an imbedding in case $2m < n$ and satisfies the following condition if $2m = n$. If $fx = fy$, where $x, y \in C$, $x \neq y$, then each of x, y is interior to an m-simplex of C_0 and $f^{-1}fx$ contains no point other than x, y. Points such as x, y will be called *singular* (with respect to f).

Assume that f is normal. Then it is an imbedding if $2m < n$. So we assume that $2m = n$. Let $x, y \in C$ be such that $x \neq y$, $fx = fy$ and let

$x \in \operatorname{int}(\sigma^m)$, where σ^m is an m-simplex of C_0. If $m = 1$ and σ^m contains more than one singular point let x be the one nearest to $P \cap \sigma^m$. Let $\sigma^{m-1} = P \cap \sigma^m$ and let A be a 1-element in σ^m which joins x to a point $z \in \operatorname{int}(\sigma^{m-1})$ and is such that int A does not meet $\dot{\sigma}^m$ or the set of singular points. We shall show how the singular points x, y can be eliminated in the way indicated by the diagram.



Let C_1, M_1 be triangulations of C, M with respect to which f is simplicial, C_1 being a triangulation of the pair (C, A) and a subdivision of C_0. Let C_1'', M_1'' be the second barycentric subdivisions of C_1, M_1 and let P_1'' be the subcomplex of C_1'' which subdivides P_0. Clearly x, y, z are vertices of C_1 and f is simplicial with respect to C_1'', M_1''. Let

$$E_0^m = N(A, C_1'') , \qquad E_1^m = N(y, C_1'') , \qquad E^n = N(fA, M_1'') .$$

The maps $f \mid A$, $f \mid E_i^m$ are imbeddings and it follows from (2.6) that E_i^m, hence also fE_i^m, and E^n are elements. Moreover $E_0^m \cap P_1''$ is the $(m-1)$-element $N(z, P_1'')$. Let $E^{m-1} = \dot{E}_0^m - \operatorname{int} N(z, P_1'')$. Then E^{m-1} is an $(m-1)$-element, $fE_i^m \subset E^n$ and

$$\dot{E}^n \cap fE_0^m = fE^{m-1} , \qquad \dot{E}^n \cap fE_1^m = f\dot{E}_1^m \subset \dot{E}^n - fE^{m-1} .$$

Let E_1^n be the second barycentric subdivision of the complex E^n and let

$$E_2^n = N(fE_0^m, E_1^n) , \qquad E^{n-1} = E^n \cap \dot{E}_2^n = N(fE^{m-1}, \dot{E}_1^n) .$$

Then E_2^n, E^{n-1} are regular neighbourhoods in E^n, $\dot{E}^n$ of fE_0^m, fE^{m-1}. Therefore they are elements. Let

$$E_3^n = E^n - \operatorname{int} E_2^n - \operatorname{int} E^{n-1} \subset E^n - fE_0^m .$$

Then E_3^n is an n-element, by (2.2), and $f\dot{E}_1^m \subset \dot{E}_3^n$. Therefore $f \mid \dot{E}_1^m$ can be extended to an imbedding $g\colon E_1^m \to E_3^n$. Define $f_1\colon C \to \operatorname{int} M$ by $f_1 = f$ in $C - E_1^m$, $f_1 = g$ in E_1^m. The points x, y are non-singular with respect to f_1 and no new singular points have been introduced. The other singular points, if any, can be eliminated in the same way.

It remains to prove that i has a normal extension. Let $f: C \to \text{int } M$ be any (piecewise linear) extension of i. Let $\{E_j\}$ be a finite set of n-elements in M whose interiors cover int M and let $\varphi_j: E_j \to \Delta$ be a homeomorphism on an n-simplex Δ. Let T be a subdivision of C_0 such that, for each simplex $\sigma \in T$ and some $j(\sigma)$,

$$f \operatorname{St}(\sigma, T) \subset \text{int } E_{j(\sigma)} \tag{2.8}$$

(f need not be simplicial with respect to T). Let W be the union of P, T^{k-1} and, possibly, some k-simplexes of T not in P ($0 \leqq k \leqq m$, $T^{-1} = \varnothing$). Assume that $f \mid W$ is normal in the above sense, with W, regarded as a subcomplex of T, playing the part of C_0.

Let τ be a k-simplex of T not in W and define

$$\psi: f^{-1}E_j \to \Delta \qquad (j = j(\tau))$$

by $\psi(x) = \varphi_j f(x)$. Since E_j is a polyhedron and f, φ_j are piecewise linear it follows that $f^{-1}E_j$ is a polyhedron and that ψ is piecewise linear. Therefore there is a triangulation K_j of $f^{-1}E_j$, such that ψ is barycentric in each simplex of K_j. Moreover we may assume that K_j is a subcomplex of a subdivision of T. Then it contains a subcomplex which subdivides $\operatorname{St}(\tau, T)$ and every simplex of K_j is contained in a simplex of T. We also assume that K_j has at least one vertex in int (τ). Let $b_1, \cdots, b_q$ be the vertices of K_j, ordered so that $b_1, \cdots, b_p$ are the ones in int (τ). Let $c_1, \cdots, c_p$ int Δ be points which are in general position with respect to each other and to $\psi b_{p+1}, \cdots, \psi b_q$. Let $\psi_1: f^{-1}E_j \to \Delta$ be the map, barycentric in each simplex of K_j, which is defined by $\psi_1 b_i = c_i$ or ψb_i according as $i \leqq p$ or $i > p$. If $k = m$ let $c_1, \cdots, c_p$ be such that no m-simplex of $\psi_1 K_j$ with one or more of $c_1, \cdots, c_p$ among its vertices contains a point $\psi_1\sigma_1 \cap \psi_1\sigma_2$, where σ_1 and σ_2 are disjoint closed m-simplexes of $K_j \cap (W \cup \tau)$. Define $f_1: C \to \text{int } M$ by

$$\begin{aligned} f_1 x &= \varphi_j^{-1}\psi_1 x && \text{if } x \in f^{-1}E_j \\ &= fx && \text{if } x \in C - \text{int St}(\tau, T) . \end{aligned}$$

The map $f_1 \mid W \cup \tau$ is normal and we take c_i so near to ψb_i ($i = 1, \cdots, p$) that f_1 satisfies (2.8). Then it follows inductively that i has a normal extension, $C \to \text{int } M$, and the proof is complete.

LEMMA (2,9). *Let M, P be as in* (2.7) *and let M be connected and bounded. Then M can be imbedded in $M - P$.*

PROOF. Let E be as in (2.7) and let A be a 1-element which joins a point $x \in \dot{M}$ to a point in $\dot{E}$ and does not meet $\dot{M} \cup E$ anywhere else. Let K be a triangulation of the pair $(M, A \cup E)$ and let $E_0 = N(A \cup E, K'')$. Clearly $A \cup E$ is completely collapsible. Therefore E_0 is an n-element and $\dot{M} \cap E_0$ is the $(n-1)$-element $N(x, K'')$. By (2.2), M is homeomorphic to $M - \operatorname{int} E_0 - \operatorname{int} N(x, K'')$, which is in $M - P$.

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