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A NOTE ON AN EXAMPLE OF MAZUR

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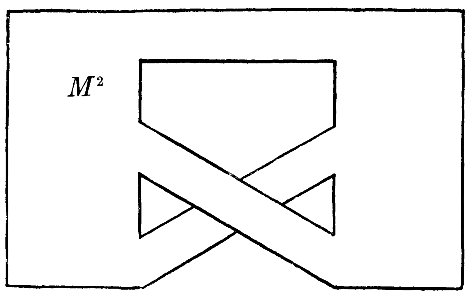
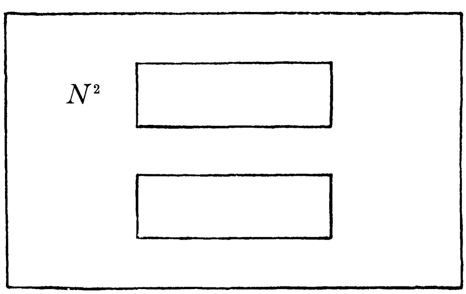
Poenaru [4], Mazur [3] and Curtis [2] have given examples of non-trivial bounded manifolds M^n such that $M^n \times I = I^{n+1}$ (where I denotes the unit interval and I^n the n-cube). These examples are temptingly just out of reach of visual imagination, because, as Bing [1] has shown, M^n must be at least 4-dimensional, and so the untwisting of $M^n \times I$ to make it into a cube must be at least 5-dimensional.

On the other hand, if we relax the condition that $M \times I$ be a cube, and merely look for two different manifolds M, N such that $M \times I = N \times I$, then as Whitehead [5] observed we can go right down to 2-dimensions, and choose M^2 to be a torus with a hole, and N^2 a disk with two holes. We show that the construction of Whitehead's 2-dimensional example can be made to echo word for word that of Mazur's 4-dimensional example, and consequently the easy untwisting of the former gives intuitive insight into the more difficult untwisting of the latter.





Mazur's example [3]

Start with $S^1 \times I^3$. In the boundary $S^1 \times S^2$, choose a knotted S^1 homologous to the first factor. Knotted means that S^1 is not isotopic to a 1-sphere $S^1 \times y$, $y \in S^2$. Form M^4 from $S^1 \times I^3$ by attaching a handle to S^1 (i.e., attach a disk to S^1, and then fatten the disk so that its fattened boundary is identified with some chosen tubular neighbourhood of S^1 in $S^1 \times S^2$). Form the cube I^4 by the same process, only omitting the knotting. The knotting ensures that $M^4 \neq I^4$. But one extra dimension permits unknotting $M^4 \times I = I^4 \times I$ (by just untwisting the handle).

Whitehead's example [5]

Start with $S^0 \times I^2$. In the boundary $S^0 \times S^1$, choose three linked S^0's, each homologous to the first factor. Linked means that the S^0's are not isotopic to a set of three 0-spheres $S^0 \times y_i$, $y_i \in S^1$, $i = 1, 2, 3$. Orient the two components of $S^0 \times I^2$, by regarding them as part of the oriented boundary of $I \times I^2$. Form M^2 from $S^0 \times I^2$ by attaching three oriented handles to the three S^0's (i.e., attach an arc to each S^0, and then fatten the arc so that its fattened end points are identified with some chosen neighbourhood of S^0 in $S^0 \times S^1$, in such a way that the handle can be oriented to agree with the orientation of $S^0 \times I^2$). Form N^2 by the same process, only omitting the linking. The linking ensures that $M^2 \neq N^2$. But one extra dimension permits unlinking $M^2 \times I = N^2 \times I$ (by just untwisting the handles).

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