DIHOMOLOGY
III. A GENERALIZATION OF THE POINCARÉ DUALITY FOR MANIFOLDS

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1. Introduction

THE Poincaré duality for manifolds may be generalized to arbitrary topological spaces. The result is a spectral sequence E running

$$H^q(X;\, \mathfrak{L}_p) \overset{q}{\Longrightarrow} H_{p-q}(X),$$

which is a topological invariant of the space X, but not an invariant of homotopy type. The E^2 term is the Čech cohomology of X with coefficients in the sheaf $\mathfrak{L}$ of local singular homology of X, and the E^∞ term is related to the ordinary global singular homology of X. The sequence therefore relates the local and global structures of the space.

If X is a closed orientable n-manifold, then the local homology sheaf reduces to the simple sheaf of integers, and the spectral sequence collapses to the familiar isomorphism of Poincaré, $H^q \cong H_{n-q}$.

If X is a polyhedron, there is a simple way of defining E, and its dual $\hat{E}$, using a triangulation. In order to obtain the spectral sequences as quickly as possible we give this simplicial method in § 2, and summarize the properties of the sequences in Theorem 1.

In § 3 we generalize $E, \hat{E}$ to arbitrary topological spaces, using a combination of singular homology and Čech cohomology, and verify that the simplicial sequences are in fact topological invariants. The functorial qualities of $E, \hat{E}$ are also discussed. Since both E and $\hat{E}$ involve both homology and cohomology, they are not functors on the category of topological spaces but we prove in Theorem 2 that if we generalize them further they are functors on a category of maps, the sequences of a space being those associated with the identity map.

The last section is concerned with the geometrical interpretation of E and $\hat{E}$, and, in particular, of the filtrations induced on the homology and cohomology groups. It appears that the filtration of a homology or cohomology class has something to do with the dimension of that part of the space in which it is 'situated', and we prove two theorems which are feelers in this direction. Theorem 3 connects the homology filtration with cap products, while Theorem 4 concerns the filtration of a cohomology class. Define the *codimension* of a cohomology class to be the minimum dimension of the support of a cocycle in that class. Theorem 4 shows that codimension $\geqslant$ filtration, and we conjecture that on polyhedra codimension = filtration.

The paper uses definitions and techniques introduced in (**5**), but is self-contained; the main difference between this paper and (**5**) is the mixture here between homology and cohomology. Theorems 1, 2, and 3 were contained in a thesis submitted for a doctorate at Cambridge in 1954, and Theorem 1 was stated in (**3**). The sequence E was used in its collapsed form in (**1**) to prove the Poincaré duality between the singular homology and the Čech cohomology of a topological manifold. An analogous spectral sequence, defined using local Čech homology, which is isomorphic to E on polyhedra, was discovered independently by I. Fary in 1955.

Contents

2. Simplicial theory

Let K be a finite oriented simplicial complex of dimension n. We use the same symbol K also to denote the resulting geometric chain complex $K = \Sigma K_p$. The purpose of this section is to define the spectral sequences $E(K)$ and $\hat{E}(K)$, and to establish their main properties in Theorem 1. The reader is warned that at first sight the mixture of homology and cohomology may seem unnatural, but in the next section we generalize the procedure and show that the algebra is functorial.

The dichain complex $\hat{D}$

Let $\hat{D}^p_q$ be the free Abelian group generated by all pairs of simplexes (σ^p, τ_q), where σ^p is an oriented p-simplex of K, and τ_q a q-dimensional face of σ^p. The reason for the position of the suffixes is that we regard τ_q as an elementary chain, and σ^p as an elementary cochain (mapping σ_p to 1 and other simplexes to 0). Let $\hat{D} = \sum_{p,q} \hat{D}^p_q$, and define skew-commutative differentials $\hat{d}_1, \hat{d}_2$ on $\hat{D}$ by the formulae

$$\hat{d}_1 : \hat{D}^p_q \to \hat{D}^{p+1}_q \quad \text{given by} \quad \hat{d}_1(\sigma^p, \tau_q) = (\delta\sigma^p, \tau_q),$$
$$\hat{d}_2 : \hat{D}^p_q \to \hat{D}^p_{q-1} \quad \text{given by} \quad \hat{d}_2(\sigma^p, \tau_q) = (-)^p (\sigma^p, \partial\tau_q),$$

where the notation is distributive. Then $\hat{D}$ is a bigraded bidifferential group. We make $\hat{D}$ into a *dichain complex* by further specifying a total degree and a total differential. Define the *total degree* to be $s = p-q$, the *total grading* $\hat{D}^s = \sum_{p-q=s} \hat{D}^p_q$, and the *total differential* to be $\hat{d} = \hat{d}_1 - \hat{d}_2$. (The reason for the minus signs is explained in the next section.) Then $\hat{d}$ is a coboundary $\hat{d} : \hat{D}^s \to \hat{D}^{s+1}$, and so we can form the cohomology group $H^*(\hat{D}) = \sum_s H^s(\hat{D})$.

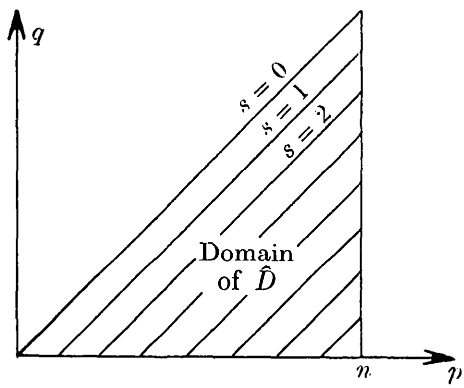

FIG. 1

The dichain complex D

Let $D^q_p = \hat{D}^p_q \pitchfork Z = \text{Hom}(\hat{D}^p_q, Z)$, the group of homomorphisms of $\hat{D}^p_q$ into the integers Z. Let $D = \hat{D} \pitchfork Z = \sum_{p,q} D^q_p$. Then D is also a dichain complex, and since K is finite, both D and $\hat{D}$ are free and finitely generated. The total degree of D is $s = p-q$, the total grading is given by $D_s = \sum_{p-q=s} D^q_p$, and the total differential by $d = \hat{d} \pitchfork 1$. Then d is a boundary $d : D_s \to D_{s-1}$, and so we can form the homology group $H_*(D) = \sum_s H_s(D)$.

The spectral sequences

Define $E(K), \hat{E}(K)$ to be the spectral sequences obtained from $D, \hat{D}$, respectively, by filtering with respect to q.

Notation for spectral sequences

In order to decide whether the suffixes p, q, r, s should be subscripts or superscripts in the two spectral sequences $E, \hat{E}$ we have the following conventions. The positions of the filtration degree q and the complementary degree p are inherited from $D, \hat{D}$; the position of the total degree s is the same as that of p because $s = p-q$; the position of the spectral index r is opposite to that of s (analogous to the usual convention that r lies opposite to $n = p+q$). Thus E is composed of terms E^r, $r = 2, 3, \dots, \infty$, where $E^r = \sum_{p,q} E^{r,q}_p$; and $\hat{E}$ is composed of terms $\hat{E}_r$, $r = 2, 3, \ldots, \infty$, where $\hat{E}_r = \sum_{p,q} \hat{E}_{r,q}^{\,p}$. The differential d^r on E^r increases the filtration degree q by r, decreases the total degree s by 1, and so increases the complementary degree p by $r-1$. The differential $\hat{d}_r$ on $\hat{E}_r$ behaves dually. Therefore

$$d^r : \hat{E}^{r,q}_p \to \hat{E}^{r,q+r}_{p+r-1}, \quad \hat{d}_r : \hat{E}_{r,q}^{\,p} \to \hat{E}_{r,q-r}^{\,p-r+1}.$$

Facets

Now that the spectral sequences have been defined, we want to describe the first and last terms, the E^2-term and the E^∞-term. For this it is convenient to use the language of (**5**). Notice that the set of pairs $\{(\sigma, \tau);\ \sigma \succ \tau\}$ is not a facing relation in the sense of (**5**), because it does not satisfy the facing condition. The difference will be analysed in the next section. Nevertheless we can speak about the *facet* of a simplex, which is the complex consisting of all those simplexes that are paired to it. The *right facet* of σ is the closure $\bar{\sigma}$ of σ, which is acyclic. The *left facet* of τ is $\operatorname{st}\tau$, the star of τ, which is an open subcomplex of K. The left facets are not in general acyclic, and their homology gives what we have been talking about as local homology.

As in (**5**) we use the term *stack* for a local coefficient system; a stack is a functor from K, regarded as a category, to the category of Abelian groups.

Define the *local homology stack*, $\mathfrak{L} = \Sigma\mathfrak{L}_p$, to be the graded contravariant stack on K given by

(i) $\mathfrak{L}_p \tau = H_p(\operatorname{st}\tau)$, and

(ii) if $\tau \succ \tau'$, then $\mathfrak{L}\tau' \to \mathfrak{L}\tau$ is the restriction homomorphism. (Note that this homomorphism does go the right way because $\operatorname{st}\tau$ is open in $\operatorname{st}\tau'$.)

Define the *local cohomology stack*, $\hat{\mathfrak{L}} = \Sigma\hat{\mathfrak{L}}^p$, to be the graded covariant stack on K given by

(i) $\hat{\mathfrak{L}}^p \tau = H^p(\operatorname{st}\tau)$, and

(ii) if $\tau \succ \tau'$, then $\hat{\mathfrak{L}}\tau \to \hat{\mathfrak{L}}\tau'$ is the inclusion homomorphism.

For example, if K is a closed orientable combinatorial n-manifold, then both $\mathfrak{L}_n$ and $\hat{\mathfrak{L}}^n$ reduce to the simple stack of integer coefficients, and $\mathfrak{L}_p = \hat{\mathfrak{L}}^p = 0$ for $p \neq n$.

LEMMA 1. *There are isomorphisms* $H_s(K) \xrightarrow{\cong} H_s(D)$ *and* $H^s(\hat{D}) \xrightarrow{\cong} H^s(K)$.

Proof. We prove the second; the first is given by the dual proof. The proof is a standard spectral-sequence argument, and resembles that of ((**5**) Theorem 1). Consider the p-filtration spectral sequence of $\hat{D}$ running

$$H^p H_q(\hat{D}) \overset{p}{\Longrightarrow} H^s(\hat{D}).$$

We can write $\hat{D} \cong \Pi\bar{\sigma}$, the direct product of the acyclic right facets, since $\hat{D}$ is free and finitely generated. Therefore

$$H_q(\hat{D}) \cong \Pi H_q(\bar{\sigma}) \cong \begin{cases} K \pitchfork Z, & q = 0, \\ 0 \;, & q \neq 0. \end{cases}$$

$$H^p H_q(\hat{D}) \cong \begin{cases} H^p(K), & q = 0, \\ 0 \;, & q \neq 0. \end{cases}$$

Therefore the spectral sequence collapses to $H^s(K) \cong H^s(\hat{D})$. We show by the arrows in the statement of the lemma the correct direction of the isomorphisms, because they are in fact induced by the augmentation of the acyclic right facets.

THEOREM 1. (1) *The spectral sequences of the finite simplicial complex K run*

$$E(K) : H^q(K; \mathfrak{L}_p) \overset{q}{\Longrightarrow} H_s(K),$$

$$\hat{E}(K) : H_q(K; \mathfrak{L}^p) \underset{q}{\Longrightarrow} H^s(K).$$

(2) *If K is of dimension n, the domain of both sequences is the triangle $0 \leqslant q \leqslant p \leqslant n$ (as shown in Fig. 1), and the sequences converge at $r = n+1$.*

(3) *The sequences are topological (although not homotopy type) invariants of the underlying polyhedron.*

(4) *If K is a closed orientable combinatorial n-manifold, then both sequences collapse to the Poincaré duality isomorphism.*

Proof. (1) We prove the result for the second sequence; the first is dual. The spectral sequence $\hat{E}(K)$ runs

$$H_q H^p(\hat{D}) \underset{q}{\Longrightarrow} H^s(\hat{D}).$$

By Lemma 1 we can replace $H^s(\hat{D})$ by $H^s(K)$. To identify the $\hat{E}_2$-term, we write $\hat{D} = \Sigma\, \mathrm{st}\, \tau$, the direct sum of the left facets. Therefore

$$H^p(\hat{D}) \cong \Sigma H^p(\mathrm{st}\, \tau) \cong \Sigma \hat{\mathfrak{L}}^p \tau,$$

which is the chain group of K with coefficients in the covariant stack $\hat{\mathfrak{L}}^p$. Therefore

$$H_q H^p(\hat{D}) \cong H_q(K; \hat{\mathfrak{L}}^p).$$

(2) The domain of both sequences is inherited from the domain of $\hat{D}$ in Fig. 1. If $r > n$, then both d^r and $\hat{d}_r$ move anything in the domain out of the domain, and so $d^r = \hat{d}_r = 0$. The convergence follows.

(3) We shall prove the topological invariance in the next section, in Theorem 2(6).

(4) If K is a closed combinatorial n-manifold, and $\tau \in K$, then $|\mathrm{st}\, \tau|$ is an open n-cell, and so $\mathrm{st}\, \tau$ has the cohomology of a closed n-cell modulo its

bounding $n-1$ sphere, namely

$$H^p(\operatorname{st}\tau) \cong \begin{cases} Z, & p = n, \\ 0, & p \neq n. \end{cases}$$

If K is orientable, all the homomorphisms of the stack $\hat{\mathfrak{L}}$ may be chosen to be the identity homomorphism $Z \to Z$, so that $\hat{\mathfrak{L}}^n$ is the simple stack of integer coefficients, while $\hat{\mathfrak{L}}^p = 0$, $p \neq n$. Therefore

$$\hat{E}_{2,q}^{\,p} \cong \begin{cases} H_q(K), & p = n, \\ 0 \quad , & p \neq n. \end{cases}$$

Since the $\hat{E}_2$-term is concentrated on the line $p = n$, the spectral sequence $\hat{E}$ collapses, $\hat{E}_2 = \hat{E}_\infty \cong H^*(K)$. Since $H_q(K)$ is the only non-zero term on the isogonal $s = n-q$, we have $H_q(K) \cong H^{n-q}(K)$, as desired. The dual proof shows that E also collapses onto the line $p = n$ to give the isomorphism $H^q(K) \cong H_{n-q}(K)$. The proof of Theorem 1 is complete, apart from the topological invariance.

EXAMPLE. We give one example in detail, to illustrate the type of sequence that can occur, and to indicate how to compute it. Since the space concerned is contractible (in fact is a cone), and since the sequence $E(K)$ is non-trivial (in fact has non-zero differentials $d^2, d^3, \ldots, d^n$), the example proves that E is not an invariant of homotopy type, for the sequence of a point is clearly trivial. A second more interesting example, in which the choice of coefficient group is significant, is given at the end of the paper.

Let M be a closed orientable combinatorial n-manifold, and let K be a cone on M with vertex v. To compute E^2, we first compute $\mathfrak{L}$. Let $Z, H_1, H_2, \ldots, H_{n-1}, Z$ be the integral homology groups of M, and let $Z, H^1, H^2, \ldots, H^{n-1}, Z$ be the integral cohomology groups of M, so that $H^r \cong H_{n-r}$. The simplexes of K are of three types: (i) the vertex v; (ii) the join $v\tau$ of v to a simplex $\tau \in M$; and (iii) a simplex $\tau \in M$. The following table gives the value of the stack $\mathfrak{L}_p$ for the three types.

p	0	1	2	3	...	n	$n+1$
$\mathfrak{L}_p(v)$	0	0	H_1	H_2	...	H_{n-1}	Z
$\mathfrak{L}_p(v\tau)$	0	0	0	0	...	0	Z
$\mathfrak{L}_p(\tau)$	0	0	0	0	...	0	0

Therefore E^2 has two types of non-zero terms, the first type situated on the p-axis due to the local homology at v:

$$E^{2,0}_{\;p} \cong H_{p-1}, \quad p = 2, 3, \ldots, n.$$

Terms of the second type are situated on the line $p = n+1$, and are due to the global cohomology of the open subcomplex $K - M$ (or equivalently

$K \bmod M$):

$$E^{2,q}_{n+1} \cong H^{q-1}, \quad q = 2, 3, \ldots, n,$$
$$E^{2,n+1}_{n+1} \cong Z.$$

For $2 \leqslant r \leqslant n$, the differential d^r is zero everywhere except for

$$d^r : E^{r,0}_{n-r+2} \xrightarrow{\cong} E^{r,r}_{n+1},$$

which is none other than the Poincaré duality isomorphism $H_{n-r+1} \xrightarrow{\cong} H^{r-1}$ for M. Eventually $E^{n+1} = E^{\infty}$, and the only non-zero term left is $E^{\infty,n+1}_{n+1} \cong Z$. This lies on the isogonal $s = 0$, and corresponds to the only non-zero homology group of K, $H_0(K) \cong Z$.

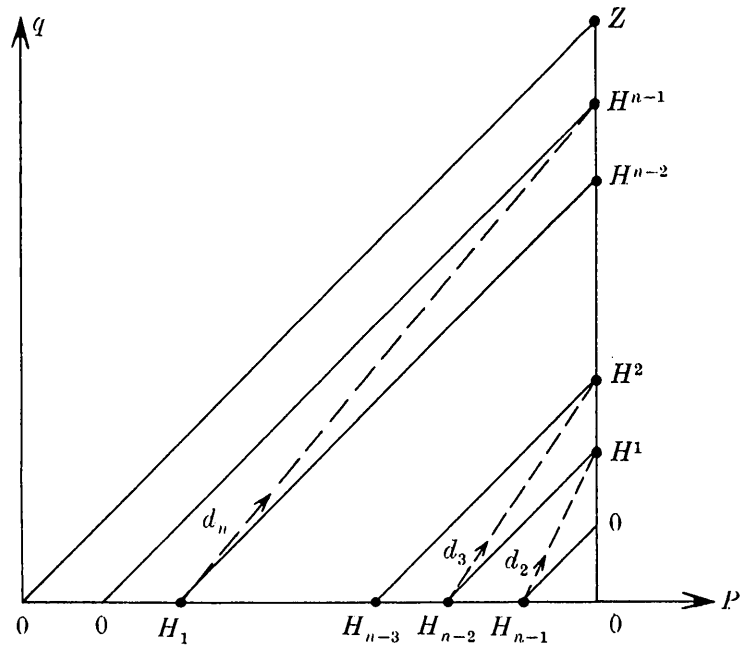

FIG. 2

3. Generalization and topological invariance

The aim of this section is to put the algebra on a proper footing, and to extend the definition of $E, \hat{E}$ to arbitrary topological spaces with arbitrary coefficients, thereby proving the topological invariance of $E(K), \hat{E}(K)$. We in fact extend the definition further, so that $E, \hat{E}$ are functors on a mixed category $\mathfrak{N}$ of continuous maps. The results are summarized in Theorem 2.

The dichain complexes $K \otimes (L \pitchfork G)$ and $K \pitchfork (L \otimes G)$

Let $K = \Sigma K_p$ and $L = \Sigma L_q$ be geometric chain complexes (for the definition see ((5) § 2)), and let G be a coefficient group. We use the symbol $\pitchfork$ in the following sense: define

$$L \pitchfork G = \sum_q L_q \pitchfork G, \quad \text{where} \quad L_q \pitchfork G = \operatorname{Hom}(L_q, G).$$

Therefore $K \otimes (L \pitchfork G)$ is a bigraded bidifferential group

$$K \otimes (L \pitchfork G) = \sum_{p,q} K_p \otimes (L_q \pitchfork G),$$

with skew commutative differentials

$$d_1 = \partial_K \otimes (1 \pitchfork 1) \quad \text{and} \quad d_2 = \omega_K \otimes (\partial_L \pitchfork 1),$$

where ∂_K, ω_K denote respectively the boundary and sign-changing automorphisms of K. We make $K \otimes (L \pitchfork G)$ into a dichain complex by defining the total degree to be $s = p - q$, and the total differential to be

$$d = d_1 - d_2 = \partial_K \otimes (1 \pitchfork 1) - \omega_K \otimes (\partial_L \pitchfork 1).$$

Dually we define the dichain complex

$$K \pitchfork (L \otimes G) = \sum_{p,q} K_p \pitchfork (L_q \otimes G),$$

with total degree again $s = p - q$, and total differential

$$\hat{d} = \hat{d}_1 - \hat{d}_2 = \partial_K \pitchfork (1 \otimes 1) - \omega_K \pitchfork (\partial_L \otimes 1).$$

Multiple mixed chain complexes

The above two examples are the only two examples of multiple mixed chain complexes that we shall use in this paper. However, it is worth mentioning the general procedure in order to explain our choice of signs and differentials. To begin with we assume that *all* boundaries and coboundaries operate on the *left*; it is necessary that they should all operate on the same side, because the total differential is composed of a mixture, and it has to operate on one side.

Let $W = W(K^1, K^2, \ldots, K^m;\, G^1, G^2, \ldots, G^n)$ be a word in the chain complexes K^i, $i = 1, 2, \ldots, m$, and the coefficient groups G^j, $j = 1, 2, \ldots, n$, formed by using the binary operators $\otimes$ and $\pitchfork$. Let $\eta^i = \pm 1$ according as to whether W is covariant or contravariant in K^i. Suppose further that W is a functor on some category (depending upon the context), and let $\eta = \pm 1$ according as to whether W is covariant or contravariant on this category. For instance, in the examples above we have regarded $K \otimes (L \pitchfork G)$ and $K \pitchfork (L \otimes G)$ as covariant and contravariant respectively.

Then we make W into a multiple chain complex as follows. The ith degree of W is given by the degree p^i of K^i. The total degree p of W is defined by the formula $\eta p = \sum_i \eta^i p^i$. The ith differential d^i of W is defined to be

$$d^i = W(\omega^1, \omega^2, \ldots, \omega^{i-1}, \partial^i, 1^{i+1}, \ldots, 1^m;\, 1, 1, \ldots, 1),$$

where ∂^i, ω^i, and 1^i denote respectively the boundary, sign-changing, and identity automorphisms of K^i, and the 1's denote the identity automorphisms of the coefficient groups. The differentials $d^1, d^2, \ldots, d^m$ are skew-commutative. The total differential d of W is defined by the formula $\eta d = \sum_i \eta^i d^i$. Then if $\eta = 1$ ($\eta = -1$) the total differential d is a boundary (coboundary) operator with respect to the total degree p, and so we can form the homology (cohomology) group $H(W)$. We can also form various spectral sequences as in ((**6**) § 3).

The rule of procedure agrees with the familiar boundary formula for cap products. For suppose cap products are written with homology on the left and cohomology on the right. Then, remembering that both boundary and coboundary operate on the left,

$$\partial(h_p \cap c^q) = \partial h_p \cap c^q - (-)^p h_p \cap \delta c^q.$$

We make use of this fact in Theorem 3 in the next section.

Carriers

Let K, L be geometric chain complexes. A *carrier* $\Gamma : K \to L$ is a function assigning to each cell $\sigma \in K$ a closed subcomplex $\Gamma\sigma$ of L, such that if $\sigma \succ \sigma'$ then $\Gamma\sigma \supset \Gamma\sigma'$. If c is a chain of K, then Γc is defined to be $\bigcup \Gamma\sigma$, the union taken over all σ that occur with non-zero coefficient in the chain c. A chain map $f : K \to L$ is *carried* by Γ if $f\sigma \in \Gamma\sigma$ for each cell $\sigma \in K$ (and therefore $fc \in \Gamma c$ for each chain $c \in K$).

We now define the *category of carriers* $\mathfrak{X}$. An *object* of $\mathfrak{X}$ is a carrier. A *map* of $\mathfrak{X}$ from the carrier $\Gamma : K \to L$ to the carrier $\Gamma' : K' \to L'$ is defined to be a pair (φ, ψ) of chain maps

$$\begin{array}{ccc} K & \xrightarrow{\ \varphi\ } & K' \\ \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma'} \\ L & \xleftarrow{\ \psi\ } & L' \end{array}$$

such that $\psi\Gamma'\varphi \subset \Gamma$. The meaning of the last inclusion sign is that $\psi\Gamma'\varphi\sigma \subset \Gamma\sigma$ for each cell $\sigma \in K$ (and therefore $\psi\Gamma'\varphi c \subset \Gamma c$ for each chain $c \in K$). The axioms for a category are easily verified, and the verification is left to the reader.

Given a carrier $\Gamma : K \to L$ we call $\Gamma\sigma$ the *right facet* (or *carrier*) of the cell $\sigma \in K$. We say that Γ is *acyclic* if all the right facets are acyclic. For instance, if K is an oriented simplicial complex, then the *identity* carrier $\Delta : K \to K$ given by $\Delta\sigma = \bar{\sigma}$ is acyclic. Returning to the carrier $\Gamma : K \to L$, we define the *left facet* of a cell $\tau \in L$ to be

$$\Gamma^{-1}\tau = \{\sigma;\ \tau \in \Gamma\sigma\},$$

which is an open subcomplex of K. Algebraically it is more correct, but less intuitive, to describe the left facet as the relative complex $(K, K-\Gamma^{-1}\tau)$. In general the left facets are not acyclic.

Carriers resemble the facing relations of (**5**), in much the same way that $\pitchfork$ resembles $\otimes$. Consequently carriers give rise to spectral sequences. Given a carrier $\Gamma: K \to L$ and a coefficient group G, it is possible to construct in a natural way 48 different spectral sequences. We select 2. The selection is uniquely determined by the two considerations:

(i) The sequences should reduce to $E(K), \hat{E}(K)$ of the previous section when Γ is the identity carrier on K and $G = Z$.

(ii) Lemmas 4 and 5 should be true.

The spectral sequences $E, \hat{E}$ of a carrier

We are given a carrier $\Gamma : K \to L$ and a coefficient group G. Construct the following two split exact sequences of dichain complexes:

$$0 \leftarrow D \leftarrow K \otimes (L \pitchfork G) \leftarrow J \leftarrow 0,$$

$$0 \to \hat{D} \to K \pitchfork (L \otimes G) \to \hat{J} \to 0,$$

where J is defined to be the subcomplex generated by

$$\{\sigma \otimes y;\ y(\Gamma\sigma) = 0\},$$

and $\hat{D}$ is defined to be the subcomplex given by

$$\hat{D} = \{x;\ x(\sigma) \in \Gamma\sigma \otimes G, \quad \text{all } \sigma \in K\}.$$

The dichain complexes $D = \Sigma D_p^q$ and $\hat{D} = \Sigma \hat{D}_q^p$ inherit their structure from the two middle terms. Define $E, \hat{E}$ to be the spectral sequences formed from $D, \hat{D}$ respectively, by filtering with respect to q.

We leave the reader to verify that in the special case when Γ is the identity carrier on a finite simplicial complex and $G = Z$, then $D, \hat{D}. E, \hat{E}$ reduce to the definitions given in § 2.

LEMMA 2. *$E, \hat{E}$ are covariant, contravariant functors, respectively, on the category of carriers $\mathfrak{X}$.*

Proof. We have already defined $E, \hat{E}$ on the objects of $\mathfrak{X}$. We must now define the functors on the maps of $\mathfrak{X}$.

A map (φ, ψ) of $\mathfrak{X}$ from the carrier $\Gamma : K \to L$ to the carrier $\Gamma' : K' \to L'$ is a pair of chain maps $\varphi : K \to K'$ and $\psi : L' \to L$, such that $\psi\Gamma'\varphi \subset \Gamma$. The induced homomorphism

$$\varphi \otimes (\psi \pitchfork 1) : K \otimes (L \pitchfork G) \to K' \otimes (L' \pitchfork G)$$

maps J to J' for the following reason. Suppose $\sigma \otimes y$ is a generator of J. Then $y\Gamma\sigma = 0$. Therefore $y\psi(\Gamma'\varphi\sigma) = y(\psi\Gamma'\varphi)\sigma \subset y\Gamma\sigma = 0$. Hence $\varphi\sigma \otimes y\psi \in J'$. But $\varphi\sigma \otimes y\psi$ is the image of $\sigma \otimes y$ under $\varphi \otimes (\psi \pitchfork 1)$. Therefore

$\varphi \otimes (\psi \pitchfork 1)$ maps J to J', and induces homomorphisms $D \to D'$ between the dichain complexes and $E \to E'$ between the spectral sequences.

Dually the homomorphism

$$\varphi \pitchfork (\psi \otimes 1) : K' \pitchfork (L' \otimes G) \to K \pitchfork (L \otimes G)$$

maps $\hat{D}'$ to $\hat{D}$. For suppose $x \in \hat{D}'$ and $\sigma \in K$. Then $\varphi\sigma \in K'$, and $x(\varphi\sigma) \in \Gamma'(\varphi\sigma) \otimes G$. Therefore

$$\begin{aligned}(\psi \otimes 1)\, x(\varphi\sigma) &\in (\psi \otimes 1)(\Gamma'\varphi\sigma \otimes G) \\ &= \psi\Gamma'\varphi\sigma \otimes G \\ &\subset \Gamma\sigma \otimes G.\end{aligned}$$

Hence $(\psi \otimes 1)\, x\varphi \in \hat{D}$. But $(\psi \otimes 1)\, x\varphi$ is the image of x under $\varphi \pitchfork (\psi \otimes 1)$. Therefore $\varphi \pitchfork (\psi \otimes 1)$ maps $\hat{D}'$ to $\hat{D}$, and induces a spectral sequence homomorphism $\hat{E}' \to \hat{E}$.

The functor axioms are easily verified, and the verification is left to the reader.

LEMMA 3. *Let $E, \hat{E}$ be the spectral sequences arising from the carrier $\Gamma : K \to L$, and $E', \hat{E}'$ those arising from $\Gamma' : K' \to L'$ (both with coefficients G). Let $\varphi : K \to K'$ be a chain map, and $\Psi : L' \to L$ an acyclic carrier, such that $\Psi\Gamma'\varphi \subset \Gamma$. Suppose ψ^1, ψ^2 are chain maps $L' \to L$ carried by Ψ. Then the two maps (φ, ψ^1) and (φ, ψ^2) from Γ to Γ' in the category $\mathfrak{X}$ induce the same spectral sequence homomorphisms $E \to E'$, $\hat{E}' \to \hat{E}$.*

Remark. A similar result holds for maps (φ^1, ψ^1) and (φ^2, ψ^2) where φ^1, φ^2 are carried by an acyclic carrier $\Phi : K \to K'$, such that $\Psi\Gamma'\Phi \subset \Gamma$. However, for the applications below we require the result stated in the lemma. The proof is analogous to that of ((**6**) Lemma 3).

Proof. We indicate the proof for $E \to E'$, and leave the dual proof to the reader. Let h be a chain homotopy between ψ^1, ψ^2 carried by Ψ. Let ω denote the sign-changing automorphism of K. Then $\varphi\omega \otimes (h \pitchfork 1)$ is a dichain homotopy between the dichain maps

$$\varphi \otimes (\psi^2 \pitchfork 1), \quad \varphi \otimes (\psi^1 \pitchfork 1) : K \otimes (L \pitchfork G) \to K' \otimes (L' \pitchfork G).$$

Also $\varphi \otimes (h \pitchfork 1)$ maps J to J', and so induces a dichain homotopy between the induced dichain maps $D \to D'$. Therefore the induced spectral sequence homomorphisms $E \to E'$ are the same (by ((**6**) Lemma 2)).

LEMMA 4. *If Γ is acyclic then the augmentation of L induces isomorphisms*

$$H_s(K; G) \xrightarrow{\cong} H_s(D), \quad H^s(\hat{D}) \xrightarrow{\cong} H^s(K; G).$$

Proof. The proof resembles that of Lemma 1 and is left to the reader; it depends upon the collapsing of the p-filtration spectral sequences. The secret lies in writing $D, \hat{D}$ in terms of the acyclic right facets, $D_p^q = \Sigma(\Gamma_q \sigma_p \pitchfork G)$ and $\hat{D}_q^p = \Pi(\Gamma_q \sigma_p \otimes G)$, where $\Gamma_q \sigma_p$ denotes the qth chain group of the right facet $\Gamma\sigma_p$, and the sum and product are taken over all p-cells $\sigma_p \in K_p$.

The left facet stacks

Lemma 4 above gives an interpretation of the ends of the spectral sequences $E, \hat{E}$. We now look at the beginnings. The left facets of Γ give rise to two stacks on L analogous to the local homology and cohomology stacks defined in the last section. Replacing $\operatorname{st}\tau$ in the definition of the local homology and cohomology stacks by $\Gamma^{-1}\tau \otimes G$ and $\Gamma^{-1}\tau \pitchfork G$, respectively, we obtain a graded contravariant stack $H_*(\Gamma^{-1}; G)$ on L, and a graded covariant stack $H^*(\Gamma^{-1}; G)$ on L.

Recall that in the definition (see ((**5**) § 2)) of a geometric chain complex $K = \Sigma K_p$, each K_p is a free Abelian group. Define K to be *of finite type* if, further, each K_p is finitely generated. For example, the nerve of a finite covering is an infinite complex of finite type.

LEMMA 5. *If either K or L is of finite type, then*

$$E_p^{2,q} \cong H^q(L; H_p(\Gamma^{-1}; G)), \quad \hat{E}_{2,q}^{\,p} \cong H_q(L; H^p(\Gamma^{-1}; G)).$$

Proof. If either K or L is of finite type, there are canonical isomorphisms (see ((**4**) Theorem 2)):

$$K \otimes (L \pitchfork G) \xrightarrow{\cong} L \pitchfork (K \otimes G),$$

$$K \pitchfork (L \otimes G) \xleftarrow{\cong} L \otimes (K \pitchfork G).$$

Therefore we can write $D, \hat{D}$ in terms of the left facets

$$D_p^q \cong \Pi(\Gamma_p^{-1}\tau_q \otimes G), \quad \hat{D}_q^p \cong \Sigma(\Gamma_p^{-1}\tau_q \pitchfork G),$$

where $\Gamma_p^{-1}\tau_q$ denotes the pth chain group of the left facet $\Gamma^{-1}\tau_q$, and the product and sum are taken over all q-cells $\tau_q \in L_q$. The lemma follows.

Remark. If neither K nor L is of finite type, then the canonical isomorphisms mentioned in the proof above become monomorphisms. Lemma 5 is no longer true, and there is no easy interpretation of the beginnings of the spectral sequences $E, \hat{E}$. If the right facets of Γ are of finite type there is a partial interpretation in terms of homology and cohomology of the 'second kind'. Selecting different definitions of $D, \hat{D}$ would get over the difficulty, but only at the expense of the more important Lemma 4. In the general case it seems to be impossible to tame both the beginnings and the ends of the spectral sequence at once.

The singular-Čech carrier of a map

We are now in a position to give the topological application of the above algebra. Let $f: X \to Y$ be a continuous map between two topological spaces. Let β be a covering of Y (all coverings are assumed to be *open*). Let $K = S(X)$ the singular simplicial complex of X. Let $L = N(\beta)$ be the nerve of the covering β. Define the *singular-Čech carrier* of the map f and the covering β to be the carrier $\Gamma: K \to L$ given by

$$\Gamma\sigma = \{\tau;\ \tau \in L \quad \text{and} \quad \operatorname{im} f\sigma \cap \sup \tau \neq \emptyset\}, \quad \sigma \in K.$$

Clearly Γ is a carrier. The right facet of a singular simplex $\sigma \in K$ is the nerve $N(\beta \mid \operatorname{im} f\sigma)$, where $\beta \mid \operatorname{im} f\sigma$ denotes the restriction of the covering β to the subset $\operatorname{im} f\sigma$ of Y. The left facet of a Čech simplex $\tau \in L$ is the relative singular complex $S(X, X - f^{-1}(\sup \tau))$. Denote the spectral sequences arising from Γ and a coefficient group G by

$$E = E(f, \beta, G), \quad \hat{E} = \hat{E}(f, \beta, G).$$

We have defined the main idea. The programme now is (i) to take limits over the directed set of coverings of Y, (ii) to show that the resulting spectral sequences are functors on a category of maps, (iii) to identify the beginnings and the ends of the sequences, and (iv) to relate them to the simplicial sequences of the last section, and to the Poincaré duality. The results are summarized in Theorem 2.

Category of maps

Define the *mixed category* $\mathfrak{N}$ *of continuous maps* as follows. An *object* of $\mathfrak{N}$ is a continuous map $f: X \to Y$ between two topological spaces. A *map* of $\mathfrak{N}$ from f to f' is a pair (φ, ψ) of continuous maps such that $\psi f' \varphi = f$.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow{\psi} & Y' \end{array}$$

We include the word *mixed* in the definition to distinguish $\mathfrak{N}$ from the category $\mathfrak{M}$ of maps introduced in (**6**), which differs from $\mathfrak{N}$ by having ψ in the above diagram going in the opposite direction. The word mixed refers to the fact that the range and domain functors on $\mathfrak{N}$ are respectively contravariant and covariant, whereas on $\mathfrak{M}$ they are both covariant.

As in (**6**), we define a larger category $\mathfrak{N}_{\text{cov}}$ which includes coverings. An *object* (f, β) of $\mathfrak{N}_{\text{cov}}$ is a continuous map $f: X \to Y$ between two topological spaces, together with a covering β of Y. A *map* $(\varphi, \psi, \psi_\beta)$ of $\mathfrak{N}_{\text{cov}}$ from (f, β) to (f', β') consists of a pair φ, ψ of continuous maps such that $\psi f' \varphi = f$, and β' refines $\psi^{-1}\beta$, together with a simplicial approximation $\psi_\beta: N(\beta') \to N(\beta)$ to ψ between the nerves of β' and β.

LEMMA 6. *$E, \hat{E}$ are covariant, contravariant functors, respectively, on the category $\mathfrak{N}_{\text{cov}}$.*

Proof. It suffices to show that the singular-Čech carrier is a covariant functor from $\mathfrak{N}_{\text{cov}}$ to the category of carriers $\mathfrak{X}$, for then we can appeal to Lemma 2. By the above definition, an object of $\mathfrak{N}_{\text{cov}}$ determines an object of $\mathfrak{X}$. We must now show that a map of $\mathfrak{N}_{\text{cov}}$ determines a map of $\mathfrak{X}$.

Suppose we are given a map $(\varphi, \psi, \psi_\beta)$ of $\mathfrak{N}_{\text{cov}}$.

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & X' \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f'} \\ Y,\beta & \xleftarrow{\ \psi,\psi_\beta\ } & Y',\beta'. \end{array}$$

Let Γ, Γ' denote the singular-Čech carriers of (f, β), (f', β'), respectively. Then (φ, ψ_β) maps Γ to Γ' in $\mathfrak{X}$ provided that $\psi_\beta \Gamma' \varphi \subset \Gamma$. To prove the last inclusion, let $\sigma \in K$, i.e. σ is a singular simplex of X. Suppose $\tau \in \psi_\beta(\Gamma'(\varphi\sigma))$, i.e. τ is some simplex in the nerve of β. Then $\tau = \psi_\beta \tau'$ for some $\tau' \in \Gamma'(\varphi\sigma)$ such that $\sup \tau \supset \psi(\sup \tau')$. Therefore

$$\begin{aligned} \operatorname{im} f\sigma \cap \sup \tau &\supset \operatorname{im} (\psi f' \varphi)\,\sigma \cap \psi(\sup \tau') \\ &\supset \psi(\operatorname{im} f' \varphi\sigma \cap \sup \tau') \\ &\neq \emptyset. \end{aligned}$$

Hence $\tau \in \Gamma\sigma$. Therefore $\psi_\beta \Gamma' \varphi \subset \Gamma$, as required. The proof that the axioms for a functor are satisfied is immediate, and is left to the reader.

LEMMA 7. *$E, \hat{E}$ are independent of approximation.*

Proof. The term independent of approximation (as defined in ((**5**) § 4)) means that any two maps $(\varphi, \psi, \psi_\beta^1)$, $(\varphi, \psi, \psi_\beta^2)$ of $\mathfrak{N}_{\text{cov}}$ from (f, β) to (f', β') with the same underlying continuous maps φ, ψ give rise to the same spectral sequence homomorphisms. Both ψ_β^1 and ψ_β^2 are carried by the acyclic Čech carrier $\Psi : L' \to L$ given by $\Psi\tau' = \{\tau;\ \sup \tau \supset \psi(\sup \tau')\}$. We verify that $\Psi \Gamma' \varphi \subset \Gamma$, as in the previous lemma, and then appeal to Lemma 3.

COROLLARY. *$E, \hat{E}$ induce functors on $\mathfrak{N}$.*

Proof. By Lemma 7 and ((**5**) Lemma 1) we can take limits. $E(f, G)$ is defined to be the direct limit of $E(f, \beta, G)$, and $\hat{E}(f, G)$ the inverse limit of $\hat{E}(f, \beta, G)$, both limits taken over the directed set of all open coverings of Y, the domain of f. Notice that $E(f, G)$ is a spectral sequence because direct limits are exact, but $\hat{E}(f, G)$ is only semi-spectral because inverse limits are only left exact.

Oriented nerves

Next we prove a lemma showing that it does not matter whether we use nerves or oriented nerves in the construction of $E, \hat{E}$ (cf. ((**6**) Lemma 4)). The lemma is useful in two contexts, to show the convergence of the spectral sequences when Y is finite-dimensional, and to relate the singular-Čech theory to the simplicial theory.

Given an object (f, β) of $\mathfrak{N}_{\text{cov}}$, where $f: X \to Y$, let $\Gamma: K \to L$ be the singular-Čech carrier, where K is the singular complex of X, and L the nerve of β. Let L^0 be an oriented nerve of β. The difference between L^0 and L is that L^0 is generated by oriented Čech simplexes, whereas L is generated by ordered Čech simplexes. The analogous carrier $\Gamma^0: K \to L^0$ is given by exactly the same formula as Γ,

$$\Gamma^0 \sigma = \{\tau;\ \tau \in L^0 \quad \text{and} \quad \operatorname{im} f\sigma \cap \sup \tau \neq \emptyset\}, \quad \sigma \in K,$$

and gives rise to analogous spectral sequences $E^0(f, \beta, G), \hat{E}^0(f, \beta, G)$, say. Let $\theta: L \to L^0$ be the natural chain equivalence from nerve to oriented nerve (see ((**2**) Chapter VI) or ((**6**) § 2)).

LEMMA 8. *The chain equivalence θ induces isomorphisms*

$$E^0(f, \beta, G) \xrightarrow{\cong} E(f, \beta, G), \quad \hat{E}(f, \beta, G) \xrightarrow{\cong} \hat{E}(f, \beta, G).$$

Proof. We prove the first and leave the dual proof to the reader. Choose one of the usual chain equivalences $\bar{\theta}: L^0 \to L$ (see ((**2**) Chapter VI) or ((**6**) § 2)). Then $\theta\Gamma = \Gamma^0$ and $\bar{\theta}\Gamma^0 \subset \Gamma$. Therefore by Lemma 2 there are induced homomorphisms between the spectral sequences

$$E^0(f, \beta, G) \underset{\bar{\theta}^*}{\overset{\theta^*}{\rightleftarrows}} E(f, \beta, G).$$

The composite $\bar{\theta}^* \theta^* = (\theta\bar{\theta})^* = 1$, because $\theta\bar{\theta} = 1$. The other composite $\theta^* \bar{\theta}^* = (\bar{\theta}\theta)^* = 1$ by Lemma 3, because both $\bar{\theta}\theta: L \to L$ and the identity on L are carried by the acyclic Čech carrier $\Psi: L \to L$ that is given by $\Psi\tau' = \{\tau;\ \sup \tau \supset \sup \tau'\}$, and which satisfies $\Psi\Gamma \subset \Gamma$. Therefore θ^* and $\bar{\theta}^*$ are isomorphisms.

Small singular simplexes

We now investigate the beginnings and the ends of the spectral sequences. We want to identify the ends with the singular homology and cohomology groups of X, but we cannot apply Lemma 4 because Γ is not in general acyclic; large singular simplexes may have complicated right facets. We therefore resort to the device of *small* singular simplexes (see ((**5**) Example vi)).

Suppose that we are given a map $f: X \to Y$ and a covering β of Y. Let α be a covering of X that refines $f^{-1}\beta$, and let $\mathbf{K} = S(X, \alpha)$ be the subcomplex of $K = S(X)$ of α-small singular simplexes. Then Γ restricted to $\mathbf{K}$ is acyclic because the right facets of α-small singular simplexes are Čech cones. Let $\mathbf{D}, \hat{\mathbf{D}}$ denote the dichain complexes formed using $\mathbf{K}$ instead of K, and let $\mathbf{E} = \mathbf{E}(f, \beta, G)$, $\hat{\mathbf{E}} = \hat{\mathbf{E}}(f, \beta, G)$ denote the resulting spectral sequences.

LEMMA 9. *The inclusion* $\mathbf{K} \subset K$ *induces isomorphisms* $\mathbf{E} \xrightarrow{\cong} E$, $\hat{E} \xrightarrow{\cong} \hat{\mathbf{E}}$.

Proof. We prove the first and leave the dual. If the covering β is finite, then the nerve L is of finite type, and there is an easy proof using Lemma 5: the inclusion induces an isomorphism between the stacks concerned, and consequently an isomorphism between the E^1-terms, $\mathbf{E}^1 \xrightarrow{\cong} E^1$. Therefore the spectral sequences are isomorphic.

If on the other hand the covering β is infinite, we form the exact sequence of dichain complexes

$$0 \to \mathbf{D} \to D \to D/\mathbf{D} \to 0.$$

We can use the standard 'chopping up' of singular simplexes to prove that the homology group of $D/\mathbf{D}$ with respect to the boundary of K vanishes. Therefore, from the relative exact homology sequence of $D, \mathbf{D}$ with respect to the boundary of K, we obtain the required isomorphism $\mathbf{E}^1 \xrightarrow{\cong} E^1$, between the E^1-terms. The lemma follows as before.

COROLLARY. *The* E^∞*-term of the spectral sequence* $E(f, \beta, G)$ *is the graded group associated with the singular homology group* $H_*(X; G)$, *suitably filtered. Dually the* $\hat{E}_\infty$*-term of* $\hat{E}(f, \beta, G)$ *is related to* $H^*(X; G)$.

Proof. The statement is true for $\mathbf{E}, \hat{\mathbf{E}}$ by Lemma 4, and is carried over to $E, \hat{E}$ by Lemma 9.

Local homology presheaves

In order to identify the beginnings of the spectral sequences, we introduce notions which generalize the local homology stacks of § 2.

Define the *local singular homology presheaf* $\mathfrak{L}$ of a space X with coefficients in a group G as follows. It is a graded presheaf $\mathfrak{L} = \Sigma \mathfrak{L}_p$, and $\mathfrak{L}_p$ is defined by assigning to each open set $U \subset X$ the relative singular homology group $\mathfrak{L}_p(U) = H_p(X, X-U; G)$, and to each pair of open sets $U \supset U'$ the inclusion homomorphism $H_p(X, X-U; G) \to H_p(X, X-U'; G)$. If, further, we are given a map $f: X \to Y$, then the image presheaf $f\mathfrak{L}$ is defined in the usual way, by assigning to each open set $V \subset Y$ the group $f\mathfrak{L}(V) = \mathfrak{L}(f^{-1} V)$, and to each pair of open sets $V \supset V'$ the homomorphism $\mathfrak{L}(f^{-1} V) \to \mathfrak{L}(f^{-1} V')$. We can form in the usual way the cohomology group $H^*(X; \mathfrak{L})$ of X with coefficients in $\mathfrak{L}$, and similarly $H^*(Y; f\mathfrak{L})$.

The *dual* notion $\hat{\mathfrak{L}} = \Sigma\hat{\mathfrak{L}}^p$ is defined by assigning to each open set $U \subset X$ the relative singular cohomology group $\hat{\mathfrak{L}}^p(U) = H^p(X, X-U; G)$, and to each pair of open sets $U \supset U'$ the (upstream) inclusion homomorphism $H^p(X, X-U'; G) \to H^p(X, X-U; G)$. Given $f: X \to Y$, the image $f\hat{\mathfrak{L}}$ is defined as above. By taking homology groups of nerves of coverings of X, with coefficients in $\hat{\mathfrak{L}}$, and by proceeding to the inverse limit, we can form the homology group $H_*(X; \hat{\mathfrak{L}})$ of X with coefficients in $\hat{\mathfrak{L}}$. Similarly we can form $H_*(X; f\hat{\mathfrak{L}})$.

Notice that both $\mathfrak{L}$ and $\hat{\mathfrak{L}}$ are different from the local singular cohomology presheaf $\mathfrak{S}$ that was used in ((**5**) § 6).

We can now state the results of this section.

THEOREM 2. (1) *$E, \hat{E}$ are functors on the mixed category of continuous maps $\mathfrak{N}$ and the category of Abelian groups. $E(f, G)$ is a spectral sequence covariant in both f and G. $\hat{E}(f, G)$ is a semi-spectral sequence (the inverse limit of spectral sequences) contravariant in f and covariant in G.*

(2) *If f maps X to Y, then $E^\infty, \hat{E}_\infty$ are related to the singular homology, cohomology groups of X, respectively. In detail there are filtrations*

$$H_s(X; G) = F^0_s \supset F^1_s \supset \ldots \supset F^q_s \supset F^{q+1}_s \supset \ldots \supset 0 \quad ,$$
$$0 \subset \hat{F}^s_0 \subset \hat{F}^s_1 \subset \ldots \subset \hat{F}^s_q \subset \hat{F}^s_{q+1} \subset \ldots \subset H^s(X; G),$$

and exact sequences

$$0 \to F^{q+1}_s \to F^q_s \to E^{\infty,q}_{q+s} \to 0,$$
$$0 \to \hat{F}^s_{q-1} \to \hat{F}^s_q \to \hat{E}_{\infty,q}^{q+s}.$$

If Y is of dimension n, then the filtrations are finite and stop at $F^{n+1}_s = 0$, $\hat{F}^s_{n+1} = H^s(X; G)$. If f is the identity map on a polyhedron of dimension n, then the filtrations stop at $F^{n-s+1}_s = 0$, $\hat{F}^s_{n-s+1} = H^s(X; G)$.

(3) *Suppose f maps X into a compact space Y. Let $\mathfrak{L}$ denote the local singular homology presheaf on X with coefficients in G, and $\hat{\mathfrak{L}}$ the dual. Then the $E^2, \hat{E}_2$ terms are given by*

$$E^{2,q}_p = H^q(Y; f\mathfrak{L}_p), \quad \hat{E}_{2,q}^{\,p} = H_q(Y; f\hat{\mathfrak{L}}^p).$$

(4) *If f is the identity map on a compact space X, then the sequences run*

$$E : H^q(X; \mathfrak{L}_p) \overset{q}{\Longrightarrow} H_s(X; G), \quad \hat{E} : H_q(X; \hat{\mathfrak{L}}^p) \underset{q}{\Longrightarrow} H^s(X; G).$$

(5) *If f is the identity map on a closed orientable topological n-manifold M, then the spectral sequences $E, \hat{E}$ collapse to the Poincaré duality isomorphisms between Čech cohomology, homology and singular homology, cohomology, respectively.*

$$\check{H}^q(M; G) \cong H_{n-q}(M; G), \quad \check{H}_q(M; G) \cong H^{n-q}(M; G).$$

(6) *If f is the identity map on a polyhedron triangulated by K, and if the integers are taken as coefficients, then $E, \hat{E}$ are isomorphic to the spectral sequences $E(K), \hat{E}(K)$ of Theorem 1. Consequently in this case $\hat{E}$ is spectral.*

Proof of Theorem 2. (1) This is the Corollary to Lemma 7.

(2) The Corollary to Lemma 9 tells us that the statements are true when the covering β is present. Therefore they remain true when we take limits. Two points should be borne in mind. Since direct limits are exact, both the filtration of $H_s(X; G)$ and the exactness of the upper short exact sequence are preserved in the limit. Inverse limits are left exact, and so the filtration of $H^s(X; G)$ is preserved, but the last zero of the lower short exact sequence is lost. The second point concerns the isomorphisms $H_s(D) \cong H_s(X; G)$, $H^s(\hat{D}) \cong H^s(X; G)$. These isomorphisms are functorial because they are induced by augmentation (Lemma 4) and inclusion (Lemma 9). Therefore they are independent of β, and so the direct, inverse systems of groups $H_s(D)$, $H^s(\hat{D})$ are systems of isomorphisms. Hence the limit groups remain isomorphic to the singular homology, cohomology groups of X, respectively.

If Y is of dimension n, we can confine ourselves to the cofinal set of coverings of dimension n. The sequences $E, \hat{E}$ for such a covering can be computed using oriented nerves by Lemma 8, and so their domains lie in the strip $0 \leqslant q \leqslant n$. Therefore $F_s^{n+1} = 0$, $F_{n+1}^s = H^s(X; G)$ for each such covering, and consequently also for the limit.

If f is the identity map on a polyhedron of dimension n, we can by (6) compute the sequences as in Theorem 1. Therefore $E, \hat{E}$ have the domain shown in Fig. 1, and the lengths of the filtrations can be read off from the lengths of the isogonals $s =$ constant.

(3) If Y is compact, we can confine ourselves to the cofinal set of finite coverings. The advantage of having β finite is that the nerve is of finite type, and so we can apply Lemma 5:

$$E^{2,q}_p \cong H^q(L; H_p(\Gamma^{-1}; G)), \quad \hat{E}_{2,q}^{p} \cong H_q(L; H^p(\Gamma^{-1}; G)).$$

Since the left facets of Γ are the relative singular complexes

$$\Gamma^{-1}\tau = S(X, X - f^{-1}(\sup \tau)), \quad \tau \in L,$$

the contravariant stack $H_p(\Gamma^{-1}; G)$ on L is none other than that arising from the presheaf $f\mathfrak{L}_p$. Taking direct limits gives the first result. The second result is dual.

(4) This is a special case of (2) and (3).

(5) In a closed topological n-manifold M we can confine ourselves to the cofinal set of finite coverings by n-cells U, such that

$$H_p(M, M - U; G) \cong \begin{cases} G, & p = n, \\ 0, & p \neq n. \end{cases}$$

Since M is orientable, the presheaf $\mathfrak{L}_n$ reduces to the simple presheaf G, and $\mathfrak{L}_p = 0$, $p \neq n$. Therefore the spectral sequence E collapses onto the line $p = n$ to the isomorphism

$$\check{H}^q(M;G) \cong E_2^{n,q} = E_\infty^{n,q} \cong H_{n-q}(M;G).$$

The sequence $\hat{E}$ also collapses onto the line $p = n$, to the dual isomorphism.

(6) We are given a finite oriented complex K triangulating the polyhedron X. Let 1 denote the identity map on X. We shall prove that the singular-Čech semi-spectral sequence $\hat{E}(1,Z)$ is isomorphic to the simplicial spectral sequence $\hat{E}(K)$ of Theorem 1. We leave the proof of the dual result to the reader.

Let β denote the star covering of K. We can identify K with an oriented nerve of β, and, by Lemma 8, use this to compute the spectral sequence $\hat{E}(1,\beta,Z)$. The computation is as follows. $S(X)$ is the singular complex of X. The dichain complex

$$\hat{D}(1,\beta,Z) \subset S(X) \pitchfork K$$

is determined by the singular-Čech carrier Γ given by

$$\Gamma\sigma = \{\tau;\ \tau \in K \text{ and } \operatorname{im}\sigma \cap |\operatorname{st}\tau| \neq \emptyset\}, \quad \sigma \in S(X).$$

Meanwhile the simplicial dichain complex

$$\hat{D}(K) \subset K \pitchfork K$$

is determined by the identity carrier Δ on K given by $\Delta\sigma = \bar{\sigma}$, $\sigma \in K$.

Choose an ordering of the vertices of K, and let $\varphi : K \to S(X)$ be the chain equivalence defined as follows: given an oriented simplex $\sigma \in K$, let h be the simplicial isomorphism of the standard simplex onto σ, mapping the vertices in the correct order, and define $\varphi\sigma = \pm h$ according as to whether or not this ordering is in the orientation class of σ. Then $\Gamma\varphi = \Delta$. Therefore $(\varphi, 1)$ maps Δ to Γ in $\mathfrak{X}$, and so by Lemma 2 induces a dichain map

$$\varphi \pitchfork 1 : \hat{D}(1,\beta,Z) \to \hat{D}(K).$$

If $\tau \in K$, φ induces an isomorphism from the singular cohomology group to the simplicial cohomology group $H^p(X, X - |\operatorname{st}\tau|) \xrightarrow{\cong} H^p(\operatorname{st}\tau)$. The resulting isomorphism between stacks gives, by Theorem 1 and Lemma 5, an isomorphism between the $\hat{E}_1$-terms, and so an isomorphism between the spectral sequences

$$\hat{E}(1,\beta,Z) \xrightarrow{\cong} \hat{E}(K).$$

It remains to prove the combinatorial invariance. Let K' be the first derived complex of K, β' its star covering, and $w : K' \to K$ a simplicial approximation to the identity. The next lemma, Lemma 10, shows that w induces an isomorphism between the $\hat{E}_2$-terms, and so an isomorphism between the spectral sequences

$$\hat{E}(1,\beta',Z) \xrightarrow{\cong} \hat{E}(1,\beta,Z).$$

Taking inverse limits over the cofinal set of star coverings of derived complexes of K, we obtain an isomorphism

$$\hat{E}(1,Z) \xrightarrow{\cong} \hat{E}(1,\beta,Z),$$

which, combined with the isomorphism above, completes the proof of Theorem 2:

$$\hat{E}(1,Z) \xrightarrow{\cong} \hat{E}(K).$$

LEMMA 10. *Combinatorial invariance. The approximation* $w : K' \to K$ *induces isomorphisms* $H^*(K\,;\,\mathfrak{L}) \xrightarrow{\cong} H^*(K'\,;\,\mathfrak{L})$, $H_*(K'\,;\,\hat{\mathfrak{L}}) \xrightarrow{\cong} H_*(K\,;\,\hat{\mathfrak{L}})$.

Proof. We prove the second result, and leave the dual proof to the reader.

To distinguish between the stars in K and K', we denote by $\operatorname{st}(\tau,K)$ the star in K of a simplex $\tau \in K$, and by $\operatorname{st}(\tau',K')$ the star in K' of a simplex $\tau' \in K'$. The underlying open sets of X are denoted by $|\operatorname{st}(\tau,K)|$, $|\operatorname{st}(\tau',K')|$.

The stacks on K and K' concerned in the lemma arise from the covariant presheaf $\hat{\mathfrak{L}}$, which is defined by means of relative singular cohomology, but since everything is polyhedral we can also interpret the stacks in terms of simplicial cohomology, or compact (Čech) cohomology:

$$\hat{\mathfrak{L}}\tau = H^*(X, X-|\operatorname{st}(\tau,K)|) \cong H^*(\operatorname{st}(\tau,K)) \cong H^*_c(|\operatorname{st}(\tau,K)|),$$
$$\hat{\mathfrak{L}}\tau' = H^*(X, X-|\operatorname{st}(\tau',K')|) \cong H^*(\operatorname{st}(\tau',K')) \cong H^*_c(|\operatorname{st}(\tau',K')|).$$

The last interpretation is perhaps the most satisfactory in the present context, because it emphasizes the nature of the stack homomorphisms, which are all induced by inclusion. For if U, V are open sets, $U \subset V \subset X$, then inclusion induces a homomorphism between the compact cohomology groups $H^*_c(U) \to H^*_c(V)$. In particular, suppose $|\tau'| \subset |\tau|$, where $\tau \in K$, $\tau' \in K'$. Then $|\operatorname{st}(\tau',K')|$ is contained in, is homeomorphic to, and is a deformation retract of, $|\operatorname{st}(\tau,K)|$, the retraction taking place radially towards some fixed point in the interior of τ'. Therefore inclusion induces an isomorphism $\hat{\mathfrak{L}}\tau' \xrightarrow{\cong} \hat{\mathfrak{L}}\tau$. In other words, all the little simplexes inside τ have the same coefficient group as τ. To express this homologically we introduce the notation: if L is a subcomplex of K, let PL denote the first derived complex of L. Then PL is a subcomplex of $PK = K'$. In particular $P\bar{\tau}, P\dot{\tau}$ are respectively the subdivided closure and boundary of τ. We have shown that if τ is q-dimensional, then

$$H_t(P\bar{\tau}, P\dot{\tau}\,;\,\hat{\mathfrak{L}}) \cong H_t(\bar{\tau},\dot{\tau}\,;\,\hat{\mathfrak{L}}) \cong \begin{cases} \hat{\mathfrak{L}}\tau, & t = q, \\ 0, & t \neq q. \end{cases}$$

Moreover, the simplicial approximation w induces this isomorphism

$$w_* : H_*(P\bar{\tau}, P\dot{\tau}\,;\,\hat{\mathfrak{L}}) \xrightarrow{\cong} H_*(\bar{\tau},\dot{\tau}\,;\,\hat{\mathfrak{L}}).$$

If $K_{(q)}$ denotes the q-skeleton of K, we have, summing over all the q-simplexes of K:

$$w_* : H_*(PK_{(q)}, PK_{(q-1)};\ \hat{\mathfrak{L}}) \xrightarrow{\cong} H_*(K_{(q)}, K_{(q-1)};\ \hat{\mathfrak{L}}).$$

But this can be interpreted as an isomorphism between the E_1 terms of two spectral sequences formed from the chain groups of K', K with coefficients in $\hat{\mathfrak{L}}$, filtered by $PK_{(q)}, K_{(q)}$, respectively. Since the filtrations are finite, we can deduce an isomorphism between the E_∞ terms, and an isomorphism between the homology groups:

$$w^* : H_*(K';\ \hat{\mathfrak{L}}) \xrightarrow{\cong} H_*(K;\ \hat{\mathfrak{L}}).$$

The proof of Lemma 8 and Theorem 2 is complete.

Remark. It is possible to generalize the above topological invariance proof to the following result: if f is the underlying continuous map of a simplicial map $g : K \to L$, we can compute the spectral sequences $E, \hat{E}$ of f by using the carrier $\Gamma : K \to L$, where $\Gamma\sigma = \overline{g\sigma}$.

4. Geometrical interpretation

Throughout this last section we shall assume that the map is the identity map on a given space X. We know that the resulting spectral sequence E, and the semi-spectral sequence $\hat{E}$, are topological invariants but not homotopy type invariants of X, and we wish to discover some geometrical interpretation of them. The first thing to discuss is the filtration induced upon the homology and cohomology groups of X, for here is an extra structure upon groups that are very familiar. In Theorem 3 below we show a relation between the homology filtration and cap products, and in Theorem 4 we explain the cohomology filtration in terms of supports.

Recall from Theorem 2(2) the notation

$$H_s(X;G) = F_s^0 \supset F_s^1 \supset \dots \supset F_q^s \supset F_s^{q+1} \supset \dots \supset \quad 0 \quad ,$$
$$0 \quad \subset \hat{F}_0^s \subset \hat{F}_1^s \subset \dots \subset \hat{F}_s^q \subset \hat{F}_{s+1}^q \subset \dots \subset H^s(X;G).$$

Define the *filtration* of an s-dimensional homology (cohomology) class ξ to be the maximum (minimum) q for which $\xi \in F_s^q$ (respectively $\hat{F}_q^s$). It was mentioned in the introduction that the filtration of a class appears to have something to do with the dimension of the piece of X in which it is 'situated'. For example, if X is an orientable n-manifold, then both the spectral sequences collapse onto the line $p = n$, and the above filtrations only have one non-trivial step. Therefore the filtration of any non-zero s-dimensional class is $n - s$.

If X is not a manifold, then an intuitive explanation runs something like this: suppose for the purposes of explanation that we can represent homology and cohomology classes of X by 'manifolds' contained in X. Then an s-dimensional class ξ has filtration $n-s$ if it 'lies in an n-manifold' in X. For example if ξ_s is a homology class that can be written as a cap product $\xi_s = \eta_n \cap \xi^{n-s}$, then ξ_s must 'lie in an n-manifold' representing η_n, and so ξ_s is of filtration $n-s$; this is the meaning of Theorem 3. On the other hand, if ξ^s is a cohomology class of filtration $n-s$, then ξ^s 'lies in an n-manifold' in X, and so can be represented geometrically by an '$(n-s)$-submanifold'; this is the meaning of Theorem 4.

THEOREM 3. *If X is a polyhedron, then* $H_{q+s} \cap H^q \subset F^q_s$.

Proof. We prove the theorem for integer coefficients; the proof for arbitrary coefficients is the same. Let K triangulate X, and consider $E(K)$. There is an exact sequence

$$0 \to J \to K \otimes (K \pitchfork Z) \to D \to 0,$$

where J is generated by $\{\sigma \otimes z;\ z(\bar{\sigma}) = 0\}$. Let

$$c : K \otimes (K \pitchfork Z) \to K$$

be the cap product homomorphism given by $c(y \otimes z) = y \cap z$, where y is a chain in K and z is a cochain in $K \pitchfork Z$. We deduce that c is a chain map satisfying $cd = \partial c$, for if $p = \dim y$, then

$$\begin{aligned} cd(y \otimes z) &= c(\partial y \otimes z - (-)^p y \otimes \delta z) \\ &= \partial y \cap z - (-)^p y \cap \delta z \\ &= \partial (y \cap z) \\ &= \partial c(y \otimes z). \end{aligned}$$

If $\epsilon : K \to K \otimes (K \pitchfork Z)$ is the chain map induced by the augmentation of the second factor K, then $c\epsilon = 1$, because

$$c\epsilon(y) = c(y \otimes 1) = y \cap 1 = y.$$

Now $cJ = 0$, because if $z(\bar{\sigma}) = 0$ then $c(\sigma \otimes z) = \sigma \cap z = 0$. Therefore ϵ and c induce chain maps

$$K \xrightarrow{\epsilon} D \xrightarrow{c} K$$

and homology homomorphisms

$$H_*(K) \xrightarrow{\epsilon_*} H_*(D) \xrightarrow{c_*} H_*(K)$$

such that $c_* \epsilon_* = 1$. But ϵ_* is an isomorphism by Lemma 1, and so c_* is the inverse isomorphism. Therefore c_* is precisely the isomorphism that induces the filtration on $H_*(K)$ from that on $H_*(D)$.

Now suppose that the homology class $\xi_s \in H_s(K)$ is a cap product $\xi_s = \eta_{q+s} \cap \zeta^q$. Represent η_{q+s} by the cycle y, and ζ^q by the cocycle z.

Then $y \otimes z$ is a d-cycle of $K \otimes (K \pitchfork Z)$ of filtration q, and its image $e(y \otimes z)$ under the epimorphism $e : K \otimes (K \pitchfork Z) \to D$ is a d-cycle of D. The homology class $[e(y \otimes z)]$ in $H_s(D)$ is of filtration $\geqslant q$, and is mapped by c_* into $[y \cap z]$, which is none other than the class $\xi_s \in H_s(K)$. Therefore ξ_s is of filtration $\geqslant q$, and Theorem 3 is proved.

EXAMPLE. A corollary to Theorem 3 is $\sum_{m \geqslant q} H_{m+s} \cap H^m \subset F_s^q$. It is pertinent to enquire whether or not the inclusion can be replaced by equality. In many cases it can (in particular in the case of oriented manifolds), but not always, as is shown by the following counterexample.

Let X be the real projective plane, and let $G = Z$. Then the spectral sequence converges, $E^2 = E^\infty$, and the only non-zero terms are

$$E^{\infty,2}_{\;\;2} \simeq Z, \quad E^{\infty,1}_{\;\;2} \simeq Z_2, \text{ the integers modulo 2.}$$

Therefore $F_1^1 = H_1 \simeq Z_2$, but $\sum_{m \geqslant 1} H_{m+1} \cap H^m = 0$, and so equality does not hold.

It is the non-orientability of X over Z that has prevented equality. If instead of Z we choose coefficients $G = Z_2$, then the Poincaré duality of X over Z_2 restores equality: $H_2 \cap H^1 = F_1^1 = H_1 \simeq Z_2$.

Codimension

If x is a singular cochain on a space X, and if $N \subset X$, let $x \mid N$ denote the restriction of x to N. The *support* of x, $\sup x$, is defined in the usual way: a point is not in $\sup x$ if and only if it has a neighbourhood N such that $x \mid N = 0$. Define the *codimension* of the cochain x to be the dimension of $\sup x$, where dimension means the usual topological dimension defined by means of coverings. Define the codimension of a cohomology class ξ to be the minimum codimension of a cocycle in ξ.

EXAMPLES. The circle, annulus, and solid torus are all of the same homotopy type, and are of dimensions 1, 2, and 3, respectively. We can represent a generator of the first cohomology group of each space by a cocycle with support a point, a line, and a disk, respectively. The corresponding codimensions will be 0, 1, and 2. If a particular meridional disk of the solid torus were shrunk to a point, then a representative cocycle could be chosen supported by this point, and so the codimension of the cohomology class would drop from 2 to 0.

LEMMA 11. *If ξ is an s-dimensional cohomology class of an n-dimensional polyhedron, then codimension $\xi \leqslant n-s$.*

Proof. Let K triangulate the polyhedron X, and let $K_{(s)}$ denote the s-dimensional skeleton of K. The *coskeleton* $K^{(n-s)}$ is defined as follows. As in ((**6**) § 5) we can generalize the notion of the dual cell of a simplex in a manifold to non-manifolds. If $\tau = a^0 a^1 \ldots a^s$ in K, define

$$\operatorname{dual}\tau = \overline{\operatorname{st}(a^0, K')} \cap \overline{\operatorname{st}(a^1, K')} \cap \ldots \cap \overline{\operatorname{st}(a^s, K')},$$

which is a subcomplex of the first derived complex K' of K, of dimension $\leqslant n-s$. Define the coskeleton

$$K^{(n-s)} = \bigcup \{\operatorname{dual}\tau;\ \dim\tau \geqslant s\},$$

which is also a subcomplex of K', of dimension $n-s$.

Let $Y = |K^{(n-s)}|$ be the underlying closed subspace of X. Then $X-Y$ is deformation-retractible (linearly) onto $|K_{(s-1)}|$, and therefore the singular cohomology group $H^s(X-Y) = 0$.

Let ξ be an s-dimensional cohomology class of X. If ξ is represented by the singular cocycle y, say, then $y|X-Y$ is the coboundary δz of some cochain z on $X-Y$. Extend z arbitrarily to X, and let $x = y - \delta z$. Then $\sup x \subset Y$, because $x|X-Y = 0$. Therefore x is a cocycle in ξ of codimension $\leqslant n-s$, which proves the lemma.

THEOREM 4. *If ξ is a cohomology class of a compact Hausdorff space X, then codimension $\xi \geqslant$ filtration ξ.* The proof is below.

COROLLARY 1. *If ξ is an s-dimensional cohomology class of filtration $\geqslant n-s$ on an n-dimensional polyhedron, then codimension ξ = filtration ξ $= n-s$.*

For codimension $\xi \geqslant$ filtration ξ, by the Theorem,

$\geqslant n-s$, by hypothesis,

$\geqslant$ codimension ξ, by Lemma 11.

COROLLARY 2. *An s-dimensional cohomology class of an orientable combinatorial n-manifold has codimension $n-s$.*

For such a class has filtration $n-s$, as observed in the introduction to § 4, and so Corollary 1 is applicable.

CONJECTURE. *If X is a polyhedron, then the inequality of Theorem 4 can be improved to equality.* This would be a complete geometrical interpretation of the cohomology filtration. Corollary 2 shows that the conjecture is true for orientable manifolds.

Proof of Theorem 4. The given cohomology class ξ implies the use of a given coefficient group G. Let k = codimension ξ. Choose a singular cocycle $x \in \xi$, with support Y of dimension k. Since the cocycle $x|X-Y$ has empty support, it cobounds. Therefore ξ is killed by the restriction homomorphism $H^*(X;G) \to H^*(X-Y;G)$, and so $\xi = \lambda\eta$, the image of some η under the relative inclusion homomorphism

$$\lambda : H^*(X, X-Y; G) \to H^*(X; G).$$

Suppose β is a given arbitrary open covering of X. The restriction of β to the subspace Y is denoted by $\beta | Y$. We construct an open covering β_1 of X, and a neighbourhood W of Y, with the properties

(i) β_1 refines β, and

(ii) $\beta_1 | W$ is of dimension $\leqslant k$ (the dimension of a covering is the dimension of its nerve). The construction of β_1 is as follows. Since Y is closed it is compact, and since Y is of dimension k we can refine $\beta | Y$ by a finite open covering γ of Y, of dimension $\leqslant k$. Since X and Y are compact Hausdorff spaces, they are normal. Therefore we can find a closed covering γ_1 of Y that is a reduction of γ (in the sense of ((**2**) 261, Lemma 3.3)). Therefore γ_1 also refines $\beta | Y$, and is of dimension $\leqslant k$, because the nerve of γ_1 is contained in that of γ. By ((**2**) 261, Lemma 3.4) there exists in X an open (in X) enlargement γ_2 of γ_1, which refines β and has a nerve isomorphic to that of γ_1 (of dimension $\leqslant k$). The sets of γ_2 cover an open neighbourhood, W_1 say, of Y. Choose a smaller open neighbourhood W of Y such that $\bar{W} \subset W_1$. Define β_1 to be γ_2 together with $\beta | X - \bar{W}$. Then β_1 is an open covering of X refining β, such that $\beta_1 | W = \gamma_2 | W$, which has dimension $\leqslant k$.

Choose now an open covering α of X with the properties

(i) α refines β_1, and

(ii) any set of α which meets Y is contained in W.

Let $K = S(X)$ the singular complex of X, and $\mathbf{K} = S(X, \alpha)$ the α-small subcomplex. Let $j : \mathbf{K} \to K$ be the inclusion homomorphism. There is an induced commutative diagram of singular cohomology with vertical isomorphisms

$$\begin{array}{ccc} H^*(X, X-Y;\, G) & \xrightarrow{\lambda} & H^*(X;\, G) \\ \cong \Big\downarrow j^* & & \cong \Big\downarrow j^* \\ H^*(X, X-Y, \alpha;\, G) & \xrightarrow{\boldsymbol{\lambda}} & H^*(X, \alpha;\, G). \end{array}$$

Let $\boldsymbol{\eta} = j^* \eta$ and $\boldsymbol{\xi} = j^* \xi = \boldsymbol{\lambda} \boldsymbol{\eta}$.

Choose oriented nerves L, L_1 of β, β_1, respectively, and construct the following dichain complexes

$$\hat{D} \subset K \pitchfork (L \otimes G),$$
$$\hat{D}_1 \subset K \pitchfork (L_1 \otimes G),$$
$$\hat{\mathbf{D}}_2 \subset \hat{\mathbf{D}}_1 \subset \mathbf{K} \pitchfork (L_1 \otimes G),$$

where $\hat{D}, \hat{D}_1$ are formed as usual from the singular-Čech carrier

$$\Gamma\sigma = \{\tau;\, \operatorname{im} \sigma \cap \sup \tau \neq \varnothing\}, \quad \sigma \in K,$$

where $\hat{\mathbf{D}}_1$ is formed using α-small singular simplexes (as in Lemma 9), and where $\hat{\mathbf{D}}_2$ is the subcomplex of $\hat{\mathbf{D}}_1$ given by

$$\hat{\mathbf{D}}_2 = \{x;\ x(\sigma) = 0 \text{ for all } \sigma \in \mathbf{K} \text{ such that } \operatorname{im}\sigma \cap Y = \emptyset\}.$$

In fact $\hat{\mathbf{D}}_2$ is none other than the relative dichain complex associated with $(X, X-Y)$, because we can identify $(\hat{\mathbf{D}}_2)^p_q = \Pi(\Gamma_q \sigma_p \otimes G)$, the direct product taken over all p-simplexes σ_p in the relative α-small singular complex $S(X, X-Y, \alpha) = S(X,\alpha)/S(X-Y,\alpha)$. Since the right facets of α-small simplexes are acyclic, we have by Lemma 4

$$H^*(\hat{\mathbf{D}}_2) \xrightarrow{\cong} H^*(X, X-Y, \alpha;\ G).$$

This isomorphism can be embedded in a commutative diagram:

$$\begin{array}{ccccccc}
H^*(\hat{\mathbf{D}}_2) & \longrightarrow & H^*(\hat{\mathbf{D}}_1) & \xleftarrow[j^*]{\cong} & H^*(\hat{D}_1) & \xrightarrow[\psi_*]{} & H^*(\hat{D}) \\
\cong\big\downarrow \boldsymbol{\epsilon}_2 & & \cong\big\downarrow \boldsymbol{\epsilon}_1 & & \cong\big\downarrow \epsilon_1 & & \cong\big\downarrow \epsilon \\
H^*(X, X-Y, \alpha;\ G) & \xrightarrow[\boldsymbol{\lambda}]{} & H^*(X, \alpha;\ G) & \xleftarrow[j^*]{\cong} & H^*(X;\ G) & \xrightarrow[1]{=} & H^*(X;\ G),
\end{array}$$

where the vertical isomorphisms $\boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_1, \epsilon_1, \epsilon$ are induced by the augmentations of L and L_1, and where the horizontal homomorphisms are induced respectively by the relative inclusion $\hat{\mathbf{D}}_2 \subset \hat{\mathbf{D}}_1$, by $j: \mathbf{K} \to K$, and by some simplicial approximation $\psi;\ L_1 \to L$. The purpose of the diagram is to trace the filtrations induced from the upper row of groups on the elements $\boldsymbol{\eta}, \boldsymbol{\xi}, \xi, \xi$ occurring in the lower row.

Now if $\sigma \in S(X, X-Y, \alpha)$ then $\operatorname{im}\sigma$ meets Y and is contained in some set of α, and so, by the construction of α, is contained in W. By the construction of β_1, the right facet $\Gamma\sigma$ in L_1 is of dimension $\leqslant k$. Consequently the domain of $\hat{\mathbf{D}}_2$ is contained in the strip $0 \leqslant q \leqslant k$. Therefore the filtration of any element in $H^*(\hat{\mathbf{D}}_2)$ is $\leqslant k$. In particular the filtration of $\boldsymbol{\eta}$, induced by $\boldsymbol{\epsilon}_2$, is $\leqslant k$. Consequently the filtration of $\boldsymbol{\xi} = \boldsymbol{\lambda\eta}$, induced by $\boldsymbol{\epsilon}_1$, is $\leqslant k$. By Lemma 9, j^* is an isomorphism not only on $H^*(X;\ G)$ but on the whole spectral sequence, and in particular upon the filtration of $H^*(\hat{D}_1)$. Therefore since $\boldsymbol{\xi} = j^*\xi$ the filtration of ξ, induced by ϵ_1, is $\leqslant k$. Finally the isomorphism ψ_* induces a homomorphism (not necessarily an isomorphism) of the filtration, and so the filtration of ξ, induced by ϵ, is $\leqslant k$.

Writing $\hat{F}^s_k(\beta, G)$ for the filtration terms induced on $H^s(X;\ G)$ by ϵ, we have shown that if ξ has dimension s and codimension k, then $\xi \in \hat{F}^s_k(\beta, G)$ for arbitrary β. Therefore $\xi \in \bigcap_\beta \hat{F}^s_k(\beta, G) = \varprojlim \hat{F}^s_k(\beta, G)$. In other words ξ is of filtration $\leqslant k$ in the filtration induced by the semi-spectral sequence $\hat{E}$ of X. Theorem 4 is proved.

We conclude the paper with an example to show how the filtration and codimension depend upon the choice of coefficient group. First we need a lemma.

LEMMA 12. *Suppose that a homomorphism $G \to G'$ between coefficient groups sends a G-cohomology class ξ into a G'-cohomology class ξ'. Then filtration $\xi' \leqslant$ filtration ξ, and codimension $\xi' \leqslant$ codimension ξ.*

Proof. The coefficient homomorphism induces a homomorphism $\hat{E} \to \hat{E}'$ between the resulting semi-spectral sequences, and a homomorphism $\hat{F}^s_q \to \hat{F}'^s_q$ between the filtration terms. Therefore if ξ^s has filtration q, then $\xi^s \in \hat{F}^s_q$, $\xi'^s \in \hat{F}'^s_q$, and so ξ' has filtration $\leqslant q$.

If ξ has codimension k, and x is a cocycle in ξ with codimension k, then the image G'-cocycle x' of x has if anything smaller support than x, and so has codimension $\leqslant k$. Therefore codimension $\xi' \leqslant k$.

EXAMPLE. The example is the quadric cone Q in complex projective 3-space. Q fails to be a real 4-manifold only at its vertex V. If K triangulates Q, then V must be a vertex of K, and the link of V is real projective 3-space, P^3. First we use integer coefficients Z, and later we use Z_2, the integers modulo 2.

To compute $\hat{E}_2$ we first compute $\hat{\mathfrak{L}}$. If $\tau \in K$, $\tau \neq V$, then $|\text{st}\,\tau|$ is a 4-cell. Meanwhile $H^p(\text{st}\,V) \cong \tilde{H}^{p-1}(lk\,V) = \tilde{H}^{p-1}(P^3)$, the reduced cohomology of P^3. Therefore the stack $\hat{\mathfrak{L}} = \Sigma \hat{\mathfrak{L}}^p$ is given by the table

p	0	1	2	3	4
$\hat{\mathfrak{L}}^p(\tau)$	0	0	0	0	Z
$\hat{\mathfrak{L}}^p(V)$	0	0	0	Z_2	Z

The global homology of Q is $H_*(Q) \cong Z, 0, Z, 0, Z$. Therefore the only non-zero terms of $\hat{E}_2$ are:

$$\hat{E}_{2,0}^{\,3} \cong Z_2, \text{ from the local cohomology at } V, \text{ and}$$

$$\hat{E}_{2,q}^{\,4} \cong Z, q = 0, 2, 4, \text{ from the global homology of } Q.$$

The only possible non-zero $\hat{d}_r$ is $\hat{d}_2$, and $\hat{d}_2 : \hat{E}_{2,2}^{\,4} \to \hat{E}_{2,0}^{\,3}$ must be the epimorphism $Z \to Z_2$ in order that $\hat{E}_3 = \hat{E}_\infty$ be related to the global cohomology of Q, $H^*(Q) \cong Z, 0, Z, 0, Z$. Therefore the only three non-zero terms of $\hat{E}_\infty$ are $\hat{E}_{\infty,q}^{\,4} \cong Z$, $q = 0, 2, 4$.

We now compute the same spectral sequence, only using Z_2 coefficients, which we denote by $\hat{E}'$. The stack $\hat{\mathfrak{L}}'$ is given by the table

p	0	1	2	3	4
$\hat{\mathfrak{L}}'^p(\tau)$	0	0	0	0	Z_2
$\hat{\mathfrak{L}}'^p(V)$	0	0	Z_2	Z_2	Z_2

Therefore the only non-zero terms of $\hat{E}'_2$ are:

$$\hat{E}'^{\,p}_{2,0} \cong Z_2,\ p = 2, 3, \text{ from the local cohomology at } V, \text{ and}$$

$$\hat{E}'^{\,4}_{2,q} \cong Z_2,\ q = 0, 2, 4, \text{ from the global homology of } Q.$$

This time $\hat{d}'_2 : \hat{E}'^{\,4}_{2,2} \to \hat{E}'^{\,3}_{2,0}$ is an isomorphism in order that $\hat{E}'_3 = \hat{E}'_\infty$ be related to the global cohomology $H^*(Q; Z_2) \cong Z_2, 0, Z_2, 0, Z_2$. Therefore the only three non-zero terms of $\hat{E}'_\infty$ are $\hat{E}'^{\,4}_{\infty,q} \cong Z_2$, $q = 0, 4$, and $\hat{E}'^{\,2}_{\infty,0} \cong Z_2$.

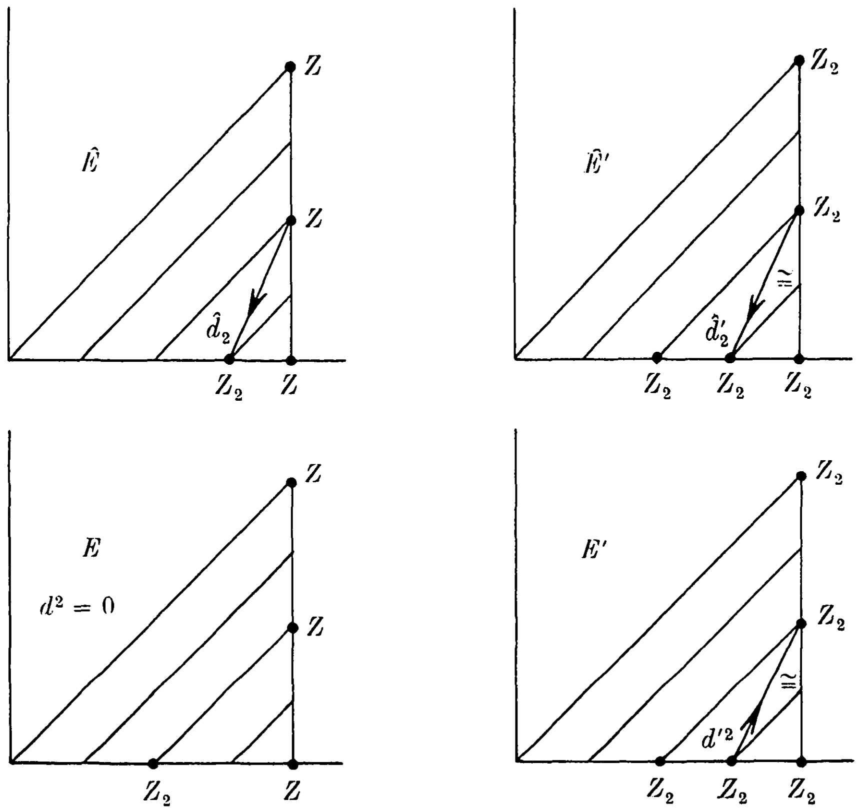

FIG. 3

The coefficient homomorphism $Z \to Z_2$ sends a generator

$$\xi \in H^2(Q) = \hat{E}^{\,4}_{\infty,2} \cong Z$$

into a generator

$$\xi' \in H^2(Q; Z_2) = \hat{E}'^{\,2}_{\infty,0} \cong Z_2.$$

Therefore

$$\text{filtration } \xi = 2, \quad \text{filtration } \xi' = 0.$$

From Corollary 1, putting $n = 4$ and $s = 2$, we obtain

$$\text{codimension } \xi = 2.$$

Therefore any integral cocycle $x \in \xi$ must have support at least 2-dimensional; the support must be some sort of surface spreading globally over Q, for example, the 2-sphere underlying a generator of the cone (a generator being a complex projective line). On the other hand, with Z_2 coefficients it is possible to find a cocycle $x' \in \xi'$ with support a single point, the vertex V. Therefore

$$\text{codimension } \xi' = 0.$$

The construction depends upon being able to find a Z_2-cocycle to represent the generator of $H^1(P^3; Z_2) = Z_2$, whose support is a real projective plane, which we then 'join' to V. The construction breaks down for integer coefficients because no such cocycle exists. The spectral sequence reflects and illuminates this interplay between local and global structures and coefficient group.

Duality

Let E, E' be the dual spectral sequences of Q with coefficients Z, Z_2, respectively. Since Z_2 is a field there is strict duality between E' and $\hat{E}'$: corresponding terms are dual vector spaces over Z_2, and corresponding differentials are conjugate maps. Therefore knowing $\hat{E}'$ we can write down E' at once.

Since Z is not a field there is no strict duality between E and $\hat{E}$. For instance, if E is also computed for Q, it turns out that $d^2 = 0$, whereas $\hat{d}_2 \neq 0$.

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