ON REGULAR NEIGHBOURHOODS

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In (**11**) J. H. C. Whitehead introduced the theory of regular neighbourhoods, which has become a basic tool in combinatorial topology. We extend the theory in three ways.

First we relativize the concept, and introduce the regular neighbourhood N of X mod Y in M, where X and Y are two compact polyhedra in the manifold M, satisfying a certain condition called link-collapsibility. We prove existence and uniqueness theorems. The idea is that N should be a neighbourhood of $X - Y$, but should avoid Y as much as possible. The notion is extremely useful in practice, and is illustrated by the following examples. We assume M to be closed for the examples.

(i) If $Y = \emptyset$ then N is a regular neighbourhood of X. Therefore the relative theory is a generalization of the absolute theory.

(ii) If X is a manifold with boundary Y, then the interior of X lies in the interior of N and the boundary of X lies in the boundary of N; in other words X is properly embedded in N.

(iii) Let X be a cone and suppose that $X \cap Y$ is contained in the base of the cone. Then N is a ball containing $X - Y$ in its interior, $X \cap Y$ in its boundary, and $Y - X$ in its exterior.

The last example was used in ((**12**) Lemma 6), and was one of the examples which suggested the need for a relative theory. Other illustrations of the use are to be found in the proofs of Theorem 2, Corollary 8, and Lemmas 7, 8, and 9 below, in the proof of Theorem 3 of (**4**), and in forthcoming papers by us on isotopy.

Secondly, Whitehead proved a uniqueness theorem that said that any two regular neighbourhoods were (piecewise linearly) homeomorphic. We strengthen this result by showing them to be isotopic, keeping a smaller regular neighbourhood fixed (Theorem 2). In fact they are ambient isotopic provided that they meet the boundary regularly (Theorem 3), which is always the case if M is unbounded.

Thirdly, Whitehead wrote the theory in the combinatorial category, and we rewrite it in the polyhedral category. The difference is that the combinatorial category consists of simplicial complexes and piecewise-linear maps, whereas the polyhedral category consists of polyhedra and piecewise-linear maps. In this paper by a *polyhedron* we mean a topological space together with a maximal non-empty family of piecewise-linearly related triangulations, each triangulation being a countable simplicial complex (for a more general definition of polyhedral space see (**14**) and (**15**)). In particular if the polyhedron is compact then each triangulation is a finite complex. The main advantage of using the polyhedral category is to be found in isotopy theory: particular triangulations need never appear in the statements of theorems, only in the proofs.

The paper is divided into three sections. In § 1 we give the definitions and state the existence and uniqueness theorems, Theorems 1, 2, and 3. Section 2 is devoted to applications, and we deduce eleven corollaries concerning spines of manifolds, knots of codimension 2, and local knottedness of embeddings and isotopies. Section 3 consists of the proofs of the three theorems.

1. Definitions and results

Notation

I stands for the unit interval.

M stands for a connected polyhedral manifold, $\dot{M}$ its boundary, and $\mathring{M}$ its interior. M may or may not be compact, and may or may not be bounded.

X, Y stand for compact polyhedra in M. $X - Y$ stands for the points in X that are not in Y. We shall always be needing two particular polyhedra obtained unsymmetrically from X and Y, and so we introduce a special notation for them:

$$X_\natural = \overline{X - Y},$$

$$Y_\natural = X_\natural \cap Y.$$

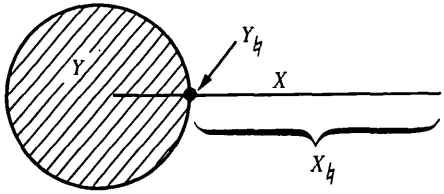

FIG. 1

We shall only use the symbol $\natural$ in contexts where it is unambiguous. Notice that if $X_{\natural\natural}, Y_{\natural\natural}$ denote the pair obtained by applying the process $\natural$ to the pair $X_\natural, Y_\natural$ then $X_{\natural\natural} = X_\natural$ and $Y_{\natural\natural} = Y_\natural$; in other words $\natural\natural = \natural$.

J, K, L stand for complexes triangulating M, X, Y. Therefore J is a countable (finite or denumerable) combinatorial manifold, and K, L are finite subcomplexes. The modulus sign $|K|$ stands for the polyhedron underlying K. Thus

$$|J| = M, \quad |K| = X, \quad |L| = Y, \quad |K_\natural| = X_\natural, \quad |L_\natural| = Y_\natural.$$

If A is a simplex of K, denote by $\mathrm{lk}(A,K)$, $\mathrm{st}(A,K)$, and $\overline{\mathrm{st}}(A,K)$, respectively, the link, open star, and closed star, of A in K.

All maps, homeomorphisms, and isotopies are piecewise linear. We use $\cong$ to denote homeomorphisms onto.

Collapsing

A complex K simplicially collapses to a subcomplex L if there exists a sequence of subcomplexes

$$K = K_0 \supseteq K_1 \supseteq \ldots \supseteq K_r = L,$$

such that for each i, $K_i - K_{i-1}$ consists of a principal simplex of K_i and a free face.

A complex K *collapses* to a subcomplex L, written $K \searrow L$, if there exist subdivisions K', L' of K, L such that K' simplicially collapses to L'. (It is not known whether or not K *simplicially* collapses to L in these circumstances.) K is *collapsible* if it collapses to a point. A polyhedron X *collapses* to a subpolyhedron Y, written $X \searrow Y$, if for some triangulation K, L of X, Y we have $K \searrow L$. X is *collapsible* if it collapses to a point.

Given two subcomplexes K, L of some larger complex, let $K_\natural, L_\natural$ be as described above. We say that K is *link-collapsible* on L if $\mathrm{lk}(A, K_\natural)$ is collapsible for each simplex A in $L_\natural$. Given two subpolyhedra X, Y of some larger polyhedron, we say that X is *link-collapsible* on Y if for some triangulation K, L of X, Y we have K link-collapsible on L.

Remark. The definitions of collapsibility and link-collapsibility for polyhedra are independent of the triangulations, the first by Theorem 7 of (**11**), and the second as follows. Suppose K^*, L^* is an arbitrary subdivision of K, L. If $A^* \in L_\natural^*$, then $\mathrm{lk}(A^*, K_\natural^*)$ is homeomorphic to the r-fold suspension of $\mathrm{lk}(A, K_\natural)$, where A is the unique simplex of $L_\natural$ whose interior contains the interior of A^*, and where $r = \dim A - \dim A^*$. Consequently K is link-collapsible on L if and only if K^* is link-collapsible on L^*.

Examples.

(i) Any polyhedron is link-collapsible on itself and on the empty set.

(ii) A simplex is link-collapsible on any subcomplex.

(iii) A manifold is link-collapsible on its boundary, and on any subpolyhedron of the boundary.

(iv) A manifold is *not* link-collapsible on an interior point.

(v) A cone is link-collapsible on its base, and on any subpolyhedron of the base.

(vi) X is link collapsible on Y if and only if $X_\natural$ is link-collapsible on $Y_\natural$.

Definition of regular neighbourhood

We rewrite Whitehead's original definition (**11**) in terms of polyhedra. Let X, N be compact polyhedra in the manifold M. We say that N is a *regular neighbourhood* of X in M if

(1) N is an m-manifold ($m = \dim M$),

(2) N is a topological neighbourhood of X in M,

(3) $N \searrow X$.

If only conditions (1) and (3) hold we say that N is a *regular enlargement* of X in M. We say that N *meets the boundary regularly* if, further,

(4) $N \cap \dot{M}$ is a regular neighbourhood of $X \cap \dot{M}$ in $\dot{M}$.

If N_1 is another regular neighbourhood of X in M, we say that N_1 is *smaller* than N if N contains a topological neighbourhood of N_1 in M.

Now the relativization. Let X, Y, N be compact polyhedra in M. We say that N is a *regular neighbourhood* of X mod Y in M if

(1) N is an m-manifold,

(2) N is a topological neighbourhood of $X - Y$ in M, and

$$N \cap Y = \dot{N} \cap Y = Y_\natural,$$

(3) $N \searrow X_\natural$.

We say that N *meets the boundary regularly* if, further,

(4) $\overline{(N \cap \dot{M}) - Y}$ is a regular neighbourhood of $X \cap \dot{M}$ mod $Y \cap \dot{M}$ in $\dot{M}$.

If N_1 is another regular neighbourhood of X mod Y in M, we say that N_1 is *smaller* than N if N contains a topological neighbourhood of $N_1 - Y$ in M.

Absolute regular neighbourhood Relative regular neighbourhood

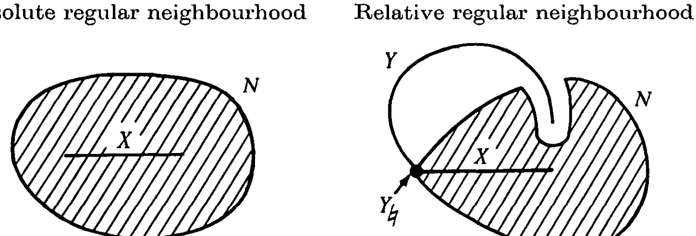

FIG. 2

Remark 1. If we put $Y = \emptyset$ in the relative definition, then we recover the absolute definition, and so the relative definition is a generalization.

Remark 2. A regular neighbourhood of X mod Y is in particular a regular enlargement of $X_\natural$.

Remark 3. Any regular neighbourhood of X mod Y is also a regular neighbourhood of $X_\natural$ mod $Y_\natural$, but not conversely in general, because of condition (2).

Remark 4. If M is unbounded then condition (4) is vacuous, and so trivially true. If M is bounded and $X \subseteq \mathring{M}$, then condition (4) is the same as saying $N \subseteq \mathring{M}$.

Remark 5. The appearance of $\overline{(N \cap \dot{M}) - Y}$ rather than $N \cap \dot{M}$ in condition (4) of the relative definition looks curious at first sight but is necessary for the following reason. Let $(X \cap \dot{M})_\natural, (Y \cap \dot{M})_\natural$ denote the pair obtained from the pair $X \cap \dot{M}, Y \cap \dot{M}$. Then $(X \cap \dot{M})_\natural \subseteq X_\natural \cap \dot{M}$, but in general they are not equal. For example consider the case when X is a manifold with boundary Y, embedded in M so that $X \cap \dot{M} = Y \cap \dot{M} = Y$. Then any regular neighbourhood of $X \cap \dot{M}$ mod $Y \cap \dot{M}$ is the empty set; but any regular neighbourhood N of X mod Y must contain Y. Therefore $N \cap \dot{M}$ cannot be a regular neighbourhood of $X \cap \dot{M}$ mod $Y \cap \dot{M}$. However, we can choose N so that it contains no more of $\dot{M}$ than Y; consequently $\overline{(N \cap \dot{M}) - Y}$ is empty, and so this choice of N will satisfy condition (4).

Second-derived neighbourhoods

Let J be a combinatorial manifold and U a subset of $|J|$. The *simplicial neighbourhood* $N(U, J)$ is defined to be the smallest closed subcomplex of J whose underlying polyhedron contains a topological neighbourhood of U in $|J|$. It consists of all closed simplexes meeting U together with their faces. In particular if K, L are subcomplexes of J, we define $N(K-L, J) = N(|K| - |L|, J)$, and deduce that $N(K-L, J) = \bigcup \overline{\text{st}}(A, J)$, the union taken over all simplexes A in $K-L$.

Suppose X, Y are polyhedra in the polyhedral manifold M. A *second-derived neighbourhood* N of X mod Y in M is constructed as follows: choose a triangulation J of M that contains subcomplexes triangulating X, Y; then choose† a second derived complex J'' of J, and define $N = |N(X-Y, J'')|$.

Isotopy

We recall the definition of isotopy (see (4)). An *isotopy of N in M* is a level-preserving embedding $f: N \times I \to M \times I$. Therefore for each t in I there is an embedding $f_t: N \to M$ such that $f(x,t) = (f_t x, t)$ for all x in N.

In the special case when $N \subseteq M$ and f_0 is the inclusion map and $N_1 = f_1 N$, we call f an *isotopy in M moving N onto N_1*. If $P \subseteq N$ and $f|P \times I$ is the identity then we say that *f keeps P fixed.*

† To form a derived complex it is not necessary to use barycentres; we can star each simplex at an arbitrary interior point. Therefore a choice is involved.

An *ambient isotopy of* M is a level-preserving homeomorphism onto, $f: M \times I \to M \times I$, such that f_0 is the identity. If $N, P \subseteq M$, $N_1 = f_1 N$, and $f | P \times I$ is the identity, then we say that f is an *ambient isotopy of* M *moving* N *onto* N_1 *and keeping* P *fixed.*

We can now state the main theorems. Let X, Y be polyhedra in M.

THEOREM 1 (Existence). *If* X *is link-collapsible on* Y *then any second-derived neighbourhood* N *of* X *mod* Y *in* M *is regular. If, further,* $X \cap \dot{M}$ *is link-collapsible on* $Y \cap \dot{M}$ *then* N *meets the boundary regularly.*

THEOREM 2 (Uniqueness). *Suppose that* X *is link-collapsible on* Y, *and let* N_1, N_2 *be two regular neighbourhoods of* X *mod* Y *in* M. *Then there exists a smaller regular neighbourhood* N_3 *and a homeomorphism of* N_1 *onto* N_2 *keeping* N_3 *fixed. Further, the homeomorphism can be realized by an isotopy in* M *moving* N_1 *onto* N_2 *through a continuous family of regular neighbourhoods and keeping* N_3 *fixed.*

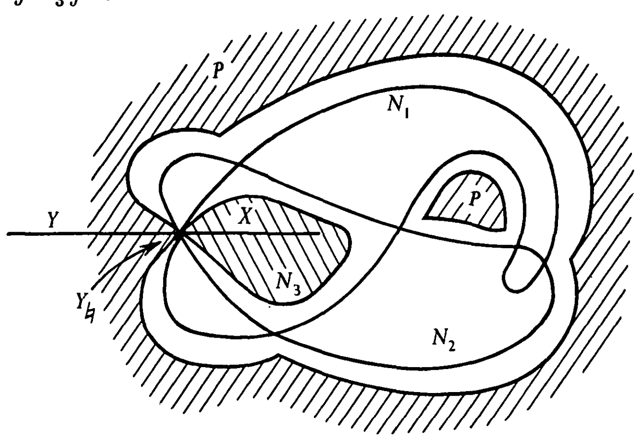

FIG. 3

If the neighbourhoods meet the boundary regularly we can strengthen the isotopy to an ambient isotopy:

THEOREM 3 (Uniqueness). *Suppose that* X *is link-collapsible on* Y, *and that* $X \cap \dot{M}$ *is link-collapsible on* $Y \cap \dot{M}$. *Let* N_1, N_2, N_3 *be three regular neighbourhoods of* X *mod* Y *in* M *meeting the boundary regularly, and such that* N_3 *is smaller than* N_1 *and* N_2. *Let* P *be the closure of the complement of a second-derived neighbourhood of* $N_1 \cup N_2$ *mod* Y *in* M. *Then there exists an ambient isotopy of* M *moving* N_1 *onto* N_2 *and keeping* $N_3 \cup P$ *fixed.* (See Fig. 3.)

Remarks.

(i) In Theorem 3, since the isotopy is ambient and keeps $X \cup Y$ fixed, it is a corollary that it moves N_1 onto N_2 through a continuous family of regular neighbourhoods that meet the boundary regularly.

(ii) Our proof of Theorem 3 below shows that in fact the ambient isotopy is by linear moves (see (4)).

(iii) It is necessary to have N_1 and N_2 meeting the boundary regularly for Theorem 3 to be true. For example suppose $Y = \emptyset$ and $X \subseteq N_1 \subseteq \mathring{M}$, and suppose N_2 meets the boundary $\dot{M}$. Then N_1 meets the boundary regularly (i.e. not at all) but N_2 does not, and it is impossible to ambient-isotope N_1 onto N_2.

(iv) It is necessary to have the smaller neighbourhood N_3 in the thesis of Theorem 2 rather than in the hypothesis (as it is in Theorem 3). For consider the following example. Let $Y = \emptyset$, and let X be a point inside a 3-ball M. Let $N_1 = M$ itself. Let N_2, N_3 be second- and third-derived neighbourhoods of a knotted arc in M, that contains X and has its end-points in $\dot{M}$. Then N_1, N_2, N_3 are regular neighbourhoods of X in M, and N_3 is smaller than N_1 and N_2. But there is no homeomorphism of N_1 onto N_2 keeping N_3 fixed, because $N_1 - N_3$ is not homeomorphic to $N_2 - N_3$. It is true that we have a situation in which N_1 and N_2 do not meet the boundary regularly, but then Theorem 2 is tailored for just such a situation. If we rechoose N_3 to be a little ball about X, then, keeping this new N_3 fixed, we can isotope N_2 off the boundary, unknot it, and then push it out onto N_1.

2. Applications

We postpone the proofs of Theorems 1, 2, and 3 until the next section, and devote this section to applications, in the form of eleven corollaries. The first seven depend only on the corresponding absolute theorems (when $Y = \emptyset$) and are concerned primarily with spines of manifolds. The last four corollaries depend essentially upon the relative theorems, and are concerned with knots of codimension 2, and with local knottedness of embeddings and isotopies. We also ask six questions concerning knots.

COROLLARY 1. THE REGULAR-NEIGHBOURHOOD ANNULUS THEOREM.
Let N, N_1 be two regular neighbourhoods of X in M. Suppose that N is smaller than N_1, and that N meets the boundary regularly. Then $\overline{N_1 - N}$ is homeomorphic to $\mathrm{Fr}(N) \times I$, *where* $\mathrm{Fr}(N)$ *denotes the frontier of N in M. In particular, if $X \subseteq \mathring{M}$ then $\overline{N_1 - N}$ is homeomorphic to $\dot{N} \times I$.*

Proof. Choose a triangulation J of M containing X as a subcomplex. Let J' be the first barycentric derived of J. Let $f : J' \to I$ be the simplicial map that maps a vertex to 0 or 1 according to whether or not it lies in X. Given ϵ, $0 < \epsilon < \frac{1}{2}$, we can choose second- and third-derived complexes of

J such that

$$N(X, J'') = f^{-1}[0, \tfrac{1}{2}] = N_2, \quad \text{say},$$
$$N(X, J''') = f^{-1}[0, \epsilon] = N_3, \quad \text{say}.$$

Then $\overline{N_2 - N_3}$ is homeomorphic to $\mathrm{Fr}(N_3) \times I$ as follows. Let A run over the simplexes of J' meeting X but not contained in X, in some order of increasing dimension. The homeomorphism is constructed inductively on $A \cap \overline{N_2 - N_3}$, which is a 'skew' prism, with walls $\dot{A} \cap \overline{N_2 - N_3}$, top $A \cap \mathrm{Fr}(N_2)$, and bottom $A \cap \mathrm{Fr}(N_3)$. By induction the homeomorphism has already been defined on the walls, so extend it to the top, bottom, and interior, each of which is a convex linear cell, by mapping some interior point arbitrarily and joining linearly to the boundary.

By Theorem 2 there is a homeomorphism $N_2 \to N_1$ keeping a smaller regular neighbourhood fixed, and therefore keeping N_3 fixed if we choose ϵ sufficiently small. Since N and N_3 both meet the boundary regularly, and are both smaller than N_1, we can ambient-isotope N_3 onto N keeping outside—N_1 fixed by Theorem 3. Therefore there are homeomorphisms

$$\overline{N_1 - N} \cong \overline{N_2 - N_3} \cong \mathrm{Fr}(N_3) \times I \cong \mathrm{Fr}(N) \times I,$$

the last because the ambient isotopy moves $\mathrm{Fr}(N_3)$ onto $\mathrm{Fr}(N)$.

If $X \subseteq \mathring{M}$ then the hypothesis that N meets the boundary regularly implies that $N \subseteq \mathring{M}$ also, and so $\mathrm{Fr}(N) = \dot{N}$.

Notice that in our proof we can, further, choose the homeomorphism $h : \overline{N_1 - N} \to \mathrm{Fr}(N) \times I$ so that $hx = (x, 0)$ for all x in $\mathrm{Fr}(N)$.

COROLLARY 2. THE ANNULUS THEOREM (Newman (7)).
If B^n is an n-ball in the interior of the n-ball B_1^n then $\overline{B_1^n - B^n} \cong S^{n-1} \times I$.

Proof. If x is an interior point of B^n, then both balls are regular neighbourhoods of x in B_1^n, and so the result follows from Corollary 1.

COROLLARY 3. *A ball collapses onto any collapsible polyhedron in its interior.*

Proof. Let B_1^n be the ball and X the collapsible polyhedron, and let B^n be a regular neighbourhood of X in the interior of B_1^n. Then B^n is a ball by Lemma 1 below. Therefore by Corollary 2, $B_1^n \searrow B^n \searrow X$. Notice that it is necessary that X be in the interior of the ball, otherwise the corollary is not true; for consider a knotted arc in a 3-ball with its ends in the boundary.

Spines

We want to generalize the last two corollaries from balls to manifolds. Let M be a compact bounded manifold, and let X be a polyhedron in the interior of M. We call X a *spine* of M if $M \searrow X$. The interest lies in finding a spine that is as 'minimal' and simple as possible. There is no loss of generality in assuming (for technical convenience) that a spine is in the interior of M, because we can first collapse away a collar from the boundary. There always exist spines of dimension less than M, because we can then collapse away all top-dimensional simplexes of some triangulation. The minimum dimension of a spine is an invariant of M; for example this dimension is zero if and only if M is a ball.

COROLLARY 4. *Let X be a spine of M and N a regular neighbourhood of X in $\mathring{M}$. Then $\overline{M-N} \cong \dot{M} \times I$.*

Proof. The result follows from Corollary 1, because both M and N are regular neighbourhoods of X in M.

COROLLARY 5. *Suppose that $X, Y \subseteq \mathring{M}$, and $X \searrow Y$. Then X is a spine of M if and only if Y is a spine of M.*

Proof. If X is a spine then trivially Y is also because $M \searrow X \searrow Y$. Conversely, suppose that Y is a spine. Let N be a regular neighbourhood of X in $\mathring{M}$. Then N is also a regular neighbourhood of Y because $N \searrow X \searrow Y$. Therefore by Corollary 4, $M \searrow N$, and so X is a spine of M because $M \searrow N \searrow X$.

Ambient simple homotopy type

Two polyhedra X, Y in M are of the same *ambient simple homotopy type*, or, more briefly, of the same *type*, if there exists a sequence of polyhedra $X = X_0, X_1, \ldots, X_k = Y$ in M such that

$$X_0 \nearrow X_1 \searrow X_2 \nearrow X_3 \searrow \ldots X_k;$$

i.e. each X_i is obtained from its predecessor by collapse or expansion. If X, Y lie in the interior of M we can without loss of generality assume that all the X_i also lie in the interior of M; for if not, choose a collar of M not meeting X or Y (a collar is an embedding $\dot{M} \times I \to M$ such that $(x, 0) \to x$ for each x in $\dot{M}$), and let $M \to \mathring{M}$ be the embedding that leaves the inside of the collar fixed and shrinks the collar to half its length. Then the images of the X_i give a sequence running from X to Y in the interior of M.

COROLLARY 6. *Any X in the interior of M is a spine if and only if it is of the type of a spine.*

Proof. One way is *a fortiori*. For the other way suppose that X is of the type of a spine X_0. In other words there is a sequence X_i, $0 \leqslant i \leqslant k$, in the interior of M, of collapses and expansions running from X_0 to $X = X_k$. Corollary 5 shows, by induction on i, that each X_i is a spine; therefore in particular X is a spine.

Remark 1. Corollary 6 is useful for simplifying spines. For example the spine of a connected bounded 3-manifold M^3 can be normalized in the following sense. We say a 0-dimensional complex is *normal* if it is a point; a 1-dimensional complex is *normal* if each vertex locally bounds exactly three 1-cells (the word 'locally' is to be interpreted by the convention that if a 1-cell has both ends at the vertex then this counts as the vertex locally bounding two 1-cells). A 2-dimensional complex is *normal* if each 1-cell locally bounds exactly three 2-cells, and each vertex locally bounds exactly four 1-cells and six 2-cells.

Given M^3 we can find a normal spine as follows. Choose a minimal spine K, that is to say one which is of minimal dimension, d say, and which cannot be collapsed any further. If $d = 0$ (M^3 is a ball) then K is a point. If $d = 1$ (M^3 is a handlebody) then expand each vertex of K into a little disk and collapse the disk from one face. If $d = 2$, expand each 1-cell of K like a banana and collapse from one side, and then expand each vertex like a pineapple and collapse from one face. In each case the corollary ensures that we are left with a spine, and the process described makes it normal.

Remark 2. Further theorems about spines can be obtained by using Smale's handle theory (**9**). For example it is shown in ((**15**) Chapter 9) that if M is simply connected and Y is a spine of M of codimension $\geqslant 3$, and if $X \subseteq Y$ is a homotopy equivalence, then X is also a spine of M.

Remark 3. In the next corollary we generalize the result from spines to arbitrary polyhedra in M. For simplicity we assume that M has no boundary, although a similar result holds for bounded manifolds.

COROLLARY 7. *Suppose that M is without boundary. Two compact polyhedra in M are of the same type if and only if their regular neighbourhoods are ambient isotopic.*

Proof. Suppose that X_0, X_k are of the same type, $X_0 \nearrow X_1 \searrow X_2 \nearrow \ldots X_k$. Let N_i be a regular neighbourhood of X_i. We show that N_i is ambient isotopic to N_0 by induction on i, the induction starting trivially with $i = 0$. Therefore assume $i > 0$. If i is even then $N_{i-1} \searrow X_{i-1} \searrow X_i$, and so N_{i-1}, N_i are both regular neighbourhoods of X_i. If i is odd then $N_i \searrow X_i \searrow X_{i-1}$, and so N_{i-1}, N_i are both regular neighbourhoods of X_{i-1}. In either case N_i is ambient isotopic to N_{i-1} by Theorem 3, and hence to N_0 by induction.

Conversely, suppose that X_0, X_1 have ambient-isotopic regular neighbourhoods N_0, N_1. To show that X_0, X_1 are of the same type it suffices to show that N_0, N_1 are of the same type, because X_i is of the same type as N_i, $i = 0, 1$. Let $f: M \times I \to M \times I$ be the given ambient isotopy, and let $N_\epsilon = f(N_0 \times \epsilon)$, $0 \leqslant \epsilon \leqslant 1$. Let N be a regular neighbourhood of N_0 in M. For sufficiently small $\epsilon > 0$, $f(N_0 \times [0, \epsilon]) \subseteq \mathring{N}$, and so by

((4) Addendum 1.2) there is another ambient isotopy moving N_0 to N_ϵ supported by N. Therefore the pair (N, N_ϵ) is homeomorphic to the pair (N, N_0), and so $N \searrow N_\epsilon$ because $N \searrow N_0$. Therefore N_0 is of the same type as N_ϵ by $N_0 \nearrow N \searrow N_\epsilon$. By the compactness of I, N_0 is of the same type as N_1.

Knotted balls and spheres

Recall the notation of (**13**). Suppose $q > m$. A *sphere pair* $S^{q,m}$ is a pair of spheres $S^q \supseteq S^m$. A *ball pair* $B^{q,m}$ is a pair of balls $B^q \supseteq B^m$, such that $\dot{B}^q \cap B^m = \dot{B}^m$. Call $q-m$ the *codimension* of the pair. The *standard* ball pair $\Delta^{q,m} = (\Sigma\Delta^m, \Delta^m)$, where Δ^m is the standard m-simplex and Σ denotes $(q-m)$-fold suspension. The standard sphere pair is the boundary of the standard ball pair one dimension higher. A pair is *unknotted* if it is homeomorphic to a standard pair. In (**13**) it was proved that any pair of codimension $\geqslant 3$ is unknotted. Knots can occur in codimension 2, but

QUESTION 1. *Is any pair of codimension* 1 *unknotted?*

The answer is 'yes' for $q \leqslant 3$ by (**1**); for $q > 3$ we know there is a topological unknotting by (**2**) and (**6**), but we do not know whether there is a piecewise linear unknotting. The answer to Question 1 depends upon

QUESTION 2. *If M^m is a combinatorial manifold triangulating a topological m-ball, then is M^m a combinatorial m-ball?*

We know the answer to Question 2 is 'yes' if $m \leqslant 3$ by the Hauptvermutung, and if $m \geqslant 6$ by (**9**), but our ignorance of the dimensions 4 and 5 prevents an inductive proof of Question 1.

Write $B^{q,m} \subseteq S^{q,m}$ if $B^q \subseteq S^q$ and $B^m = B^q \cap S^m$.

QUESTION 3. *If $B^{q,m} \subseteq S^{q,m}$ and $S^{q,m}$ is unknotted, then is $B^{q,m}$ unknotted?*

In codimension $\geqslant 3$ the answer is 'yes' by (**13**). In codimension 1 the question is the same as Question 1; for a 'yes' to Question 1 trivially implies a 'yes' to Question 3. Conversely a 'no' to Question 1 implies the existence of a knotted ball pair with unknotted boundary, and glueing two copies of this together by the involution on the boundary embeds this knotted ball pair in an unknotted sphere pair.

In codimension 2 the answer is 'yes' for $q = 3$ by the unique-factorization theorem of classical knot theory (see ((**3**) 140)); in other words an unknotted curve is not the sum of two knots. But for $q > 3$ the question is unsolved, and the various proofs for $q = 3$ break down in higher dimensions for the following reasons. (i) Schubert's unique-factorization proof ((**3**) 140) depends upon the genus of a knot, which has no sufficiently strong higher-dimensional analogue. (ii) Mazur's proof ((**3**) 142) works only if we know that the boundary of $B^{q,m}$ is unknotted, and then gives only a topological unknotting rather than a piecewise-linear one. (iii) The algebraic proof using van Kampen's theorem on the complement breaks down because, for $q > 3$, we do not know the answer to

QUESTION 4. *Given a ball pair $B = (B^q, B^{q-2})$ such that $\pi_1(B^q - B^{q-2}) \cong$ the integers, then is B unknotted?*

If $q = 3$ the answer is 'yes' by a theorem of Papakyriakopoulos (**8**), but if $q > 3$ the answer is not known.† If $q \geqslant 5$ and we add the additional hypothesis that B is locally unknotted then Stallings (**10**) gives a topological unknotting, but not necessarily a piecewise-linear one.

We now proceed to prove a partial answer to Question 3, which is useful for applications, for example in the three subsequent corollaries.

COROLLARY 8. *If $B^{q,m} \subseteq S^{q,m}$ are both unknotted pairs then the complementary ball pair $B_*^{q,m} = \overline{S^{q,m} - B^{q,m}}$ is also unknotted.*

Proof. The proof that we give works for all codimensions, although for codimension $\geqslant 3$ the result follows from (**13**), and for codimension 1 the result follows from the foundational lemma: if a ball contains another ball of the same dimension, and their boundaries meet in a common face, the closure of the complement is a ball. In codimension 2 we run into potential trouble near the boundary, but this kind of trouble is exactly what the concept of the relative regular neighbourhood is designed to cope with, as follows. Choose an unknotting homeomorphism $h : S^{q,m} \to (\Sigma\Delta^m, \Delta^m)$, where Σ denotes $(q-m)$-fold suspension. Let $B_0{}^q = h^{-1}(\Sigma(hB^m))$. Then B^q and $B_0{}^q$ are both regular neighbourhoods of B^m mod $B_*{}^m$ in S^q. Therefore by Theorem 3 there is an ambient isotopy of S^q moving B^q onto $B_0{}^q$ keeping $B^m \cup B_*{}^m = S^m$ fixed. The end of the isotopy throws $B_*{}^{q,m}$ onto $h^{-1}(\Sigma(hB_*{}^m), hB_*{}^m)$, which is unknotted. Hence $B_*{}^{q,m}$ is unknotted.

Local unknottedness

We recall the definitions of (**13**). Suppose that M, Q are manifolds and that M is compact. An embedding $f : M \to Q$ is called *proper* if $f^{-1}\dot{Q} = \dot{M}$. If f is proper denote by $\dot{f}$ the restriction of f to the boundaries, $\dot{f} : \dot{M} \to \dot{Q}$.

Let φ be a triangulation of f; that is to say, choose triangulations K, L of M, Q with respect to which f is simplicial, and call the simplicial map φ. In other words the diagram

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & L \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & Q \end{array}$$

† *Added in proof.* J. Levine has shown that, for locally unknotted ball pairs, with $q \geqslant 6$, if $B^q - B^{q-2}$ is a homotopy circle then the pair is unknotted; but knots do exist with $\pi_1(B^q - B^{q-2}) \cong$ the integers.

is commutative. Denote by $\dot{\varphi}$ the corresponding triangulation of $\dot{f}$. If v is a vertex of K, denote by $\overline{\text{st}}\,(v,\varphi)$ and $\text{lk}\,(v,\varphi)$ the pairs

$$\overline{\text{st}}\,(v,\varphi) = (\overline{\text{st}}\,(\varphi v, L),\ \varphi(\overline{\text{st}}\,(v,K))),$$
$$\text{lk}\,(v,\varphi) = (\text{lk}\,(\varphi v, L),\ \varphi(\text{lk}\,(v,K))).$$

We call f a *locally unknotted* embedding if for each v in K, $\overline{\text{st}}\,(v,\varphi)$ is an unknotted ball pair.

Remark 1. The definition is independent of the triangulation φ, because if all the vertex stars are unknotted then the same is true for any subdivision of K, L, and hence also true for any other triangulation.

Remark 2. By (**13**) local knotting can only occur in codimension 2 and possibly codimension 1.

Remark 3. 'Locally unknotted' implies 'proper'. For if v is an interior vertex of K, then v is interior to $\overline{\text{st}}\,(v,K)$, and so φv is interior to $\overline{\text{st}}\,(\varphi v, L)$, by the definition of ball pair. Similarly if v is a boundary vertex of K, then φv must be on the boundary of $\overline{\text{st}}\,(\varphi v, L)$ and so on the boundary of L. Therefore $\varphi^{-1}\dot{L} = \dot{K}$.

Remark 4. A sphere or ball pair is called locally unknotted if the inclusion map of the smaller in the larger is locally unknotted. If a pair is unknotted then it is locally unknotted, because we can triangulate with a standard pair. On the other hand if a pair is locally unknotted it may be (globally) knotted; for example consider the knots of classical knot theory.

COROLLARY 9. (i) *If f is locally unknotted then so is $\dot{f}$.*

(ii) *f is locally unknotted if and only if, for some triangulation φ of f, all the links* $\text{lk}\,(v,\varphi)$, $v \in \text{dom}\,\varphi$, *are unknotted.*

Proof. (i) Let $\varphi: K \to L$ triangulate f, and, given v in K, let $B = \overline{\text{st}}\,(v,\varphi)$. If f is locally unknotted then B is unknotted, and so $\dot{B}$ is an unknotted sphere pair. Therefore $\dot{B}$ is locally unknotted. If $v \in \dot{K}$ then $v \in \dot{B}$ and so $\overline{\text{st}}\,(v,\dot{B})$ is unknotted. But $\overline{\text{st}}\,(v,\dot{B}) = \overline{\text{st}}\,(v,\dot{\varphi})$. Therefore $\dot{f}$ is locally unknotted.

(ii) Suppose the links are unknotted; then the stars are unknotted because they are cones on the links. Conversely, suppose the stars are unknotted; then to prove that the links are unknotted it is necessary to use Corollary 8 as follows. With the notation of part (i), if $v \in \overset{o}{K}$ then $\text{lk}\,(v,\varphi) = \dot{B} =$ an unknotted sphere pair. If $v \in \dot{K}$, then $\text{lk}\,(v,\varphi) = \dot{B} - \text{st}\,(v,\dot{B})$, which is an unknotted ball pair by Corollary 8.

COROLLARY 10. *A locally unknotted ball is unknotted in its regular neighbourhood. More precisely, let (B^q, B^m) be a locally unknotted ball pair of codimension 1 or 2, which may be globally knotted. Let N^q be a regular neighbourhood of B^m in B^q. Then (N^q, B^m) is unknotted.*

Proof. The proof is similar to the proof of ((**13**) Lemmas 3 and 6), and we sketch it as follows. The essential new ingredient is Corollary 8, and we use this to prove, as in ((**13**) Lemma 3), that if two unknotted ball pairs meet in a common unknotted face, then their union is also unknotted. Now we proceed as in ((**13**) Lemma 6): triangulate B^q so that B^m collapses simplicially to a point, x say. Then clothe the expansion $x \nearrow B^m$ by pairs of second-derived neighbourhoods in B^q and B^m. In the expanding sequence of pairs of neighbourhoods, each step is equivalent to glueing on two little ball pairs each by a face. The local unknottedness of $B^m \subseteq B^q$ ensures that each little ball pair is unknotted, and by using Theorem 3 we can show that the face by which it is glued on is also unknotted: for if S is the boundary of the little ball pair and F the face, then F^{q-1} is a regular neighbourhood of F^{m-1} mod $\dot{F}^{m-1}$ in S^{q-1}, and since S is unknotted, we can unknot F by isotoping F^{q-1} onto a suspension of F^{m-1} keeping F^{m-1} fixed by Theorem 3 (as in the proof of Corollary 8). We begin with a little unknotted ball pair, namely the star of x, and each time we glue on a little ball pair the result remains unknotted. Therefore, by induction on the number of steps, we finish with $(N_*{}^q, B^m)$ unknotted, where $N_*{}^q$ is the second-derived neighbourhood of B^m in B^q. By Theorem 2 there is a homeomorphism $N^q \to N_*{}^q$ keeping B^m fixed, and so (N^q, B^m) is also unknotted.

Locally unknotted isotopies

As above, let M, Q be manifolds, with M compact. Let $f: M \times I \to Q \times I$ be an isotopy, that is to say a level-preserving embedding. Call f a *proper* isotopy if it is a proper embedding. If f is a proper isotopy then each level $f_t: M \to Q$ is a proper embedding. If f is a proper isotopy denote by ∂f the restriction of f to the boundaries, $\partial f: \dot{M} \times I \to \dot{Q} \times I$.

Remark. We have to use a different symbol ∂ for the boundary, because the boundary of f, *qua* isotopy, is smaller than the boundary of f, *qua* embedding; in other words $\partial f \neq \dot{f}$, although at each level $(\partial f)_t = (f_t)^{\cdot}$.

We define f to be a *locally unknotted isotopy* if

(i) at each level $f_t: M \to Q$ is a locally unknotted embedding, and

(ii) for each subinterval J of I the restriction $f_J: M \times J \to Q \times J$ of f is a locally unknotted embedding.

As in the case of embeddings we can deduce that any locally unknotted isotopy is proper.

COROLLARY 11. *If f is a locally unknotted isotopy then so is ∂f.*

Proof. Condition (i) for ∂f follows from Corollary 9(i) because $(\partial f)_t = (f_t)^{\cdot}$. To prove condition (ii), let K_1, K_2, K_3 be triangulations of

$(M \times J)^{\cdot}, \dot{M} \times J, M \times \dot{J}$, and let L_1, L_2, L_3 be triangulations of $(Q \times J)^{\cdot}$, $\dot{Q} \times J, Q \times \dot{J}$, respectively, such that the restrictions $\varphi_i : K_i \to L_i$, $i = 1, 2, 3$, of f are simplicial. By condition (ii) for f and Corollary 9(i), φ_1 is a locally unknotted embedding, and by condition (i) for f so is φ_3. We want to prove that φ_2 is a locally unknotted embedding. Let v be a vertex of K_2. If v is an interior vertex of K_2 then $\overline{\mathrm{st}}(v, \varphi_2) = \overline{\mathrm{st}}(v, \varphi_1)$, which is unknotted. If v is a boundary vertex then

$$\begin{aligned} \mathrm{lk}(v, \varphi_2) &= \overline{\mathrm{lk}(v, \varphi_1) - \mathrm{lk}(v, \varphi_3)} \\ &= (\text{unknotted sphere pair}) - (\text{unknotted ball pair}) \\ &= \text{unknotted ball pair, by Corollary 8.} \end{aligned}$$

Therefore $\overline{\mathrm{st}}(v, \varphi_2)$ is unknotted. Therefore ∂f is a locally unknotted isotopy.

The definition of locally unknotted isotopy that we have given immediately raises two questions:

QUESTION 5. *Does condition* (ii) *imply condition* (i)*?*

QUESTION 6. *If f is an isotopy and a locally unknotted embedding, then is f a locally unknotted isotopy?*

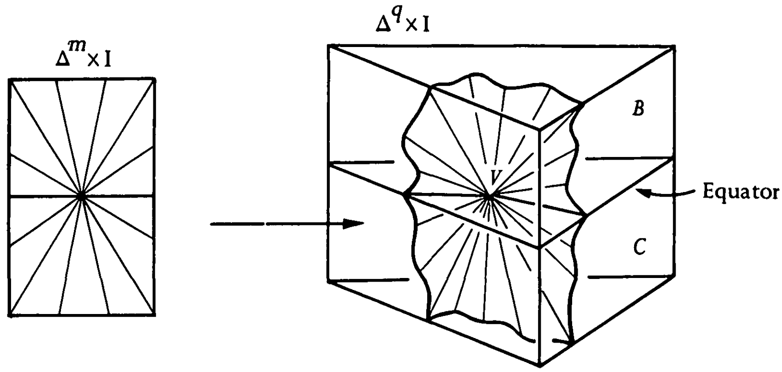

FIG. 4

If M is closed then the answer to Question 5 is 'yes' by Corollary 9(i); but if M is bounded then we may run into Question-3-type trouble at the boundary, as can be seen from the way we had to use Corollary 8 in the proof of Corollary 11. In fact if M is bounded then Question 5 is the same as Question 3; and if M is bounded or closed then Question 6 is the same as Question 3. For if the answer to Question 3 is 'yes' then it is easy to show that the answer to Questions 5 and 6 is 'yes'. Conversely, if there is a counterexample to Question 3 then we can use this counterexample to manufacture counterexamples to Questions 5 and 6. For instance, suppose B, C are knotted (q, m) ball pairs with a common boundary $\dot{B} = \dot{C}$, such that $B \cup C$ is an unknotted sphere pair. Identify B^q, C^q with the northern and southern hemispheres of the prism $\Delta^q \times I$ (i.e. those subsets of the boundary above and below the equatorial plane $t = \frac{1}{2}$; see Fig. 4). If V is the centre of the prism then $V(B \cup C)$ is unknotted, but each of VB, VC is locally knotted at V. We can regard $V(B \cup C)$ as an embedding $f: \Delta^m \times I \to \Delta^q \times I$, and by the trick of ((**4**) Lemma 4) we can make f level-preserving in a neighbourhood of the equator. The restriction of f to this neighbourhood gives a locally knotted isotopy that is a locally unknotted embedding.

The justification for the above definition of locally unknotted isotopy is that it is a sufficient condition for an isotopy to be covered by an ambient isotopy ((**4**) Theorem 2); and if f_0 is locally unknotted then it is also a necessary condition.

3. Proofs of the fundamental theorems

This last section consists of the proofs of Theorems 1, 2, and 3. Some of the lemmas come straight from (**11**), and in others we follow Whitehead's style of proof closely.

LEMMA 1. *Given $X \subseteq M$, if X is collapsible then any regular enlargement of X in M is a ball.*

Proof. Let N be the regular enlargement. By ((**11**) Theorems 4 and 7) we can choose triangulations J, K, L of M, X, N such that L collapses simplicially to K, and K collapses simplicially to a point. Then by ((**11**) Theorem 23, Corollary 1) L is a combinatorial ball, and so N is a polyhedral ball.

Full subcomplexes

Let L be a subcomplex of K. We call L a *full* subcomplex if no simplex of $K - L$ has all its vertices in L. As a consequence, any simplex of $K - L$ meets L either in a face or in the empty set.

Example (i). If L is a subcomplex of K, and K' a first-derived complex of K, then L' is full in K' (see ((**11**) Lemma 4)).

Example (ii). If L is a full subcomplex of K, and K^* an arbitrary subdivision of K, then L^* is full in K^*.

Well situated

We introduce a technical term for the convenience of the proof of Theorem 1. Let J be a combinatorial manifold and K, L finite subcomplexes. We say that the pair K, L is *well situated* in J if the following three conditions hold:

(1) K is link-collapsible on L;

(2) $K \cup L$ and $K_\natural$ are full subcomplexes of J;

(3) if A is a simplex in $N(K-L, J)-K$, then $\operatorname{lk}(A, J)$ meets K in a simplex (possibly the empty simplex).

We say that K, L is *well situated at the boundary* if, in addition,

(4) the pair $K \cap \dot{J}, L \cap \dot{J}$ is well situated in $\dot{J}$,

(5) $K \cup \dot{J}$ is a full subcomplex of J.

LEMMA 2. *Suppose that $K, L \subseteq J$, and let $N = N(K-L, J)$. If K, L is well situated in J then $N \searrow K_\natural$. If, further, K, L is well situated at the boundary then $N \searrow N \cap \dot{J} \cup K_\natural \searrow K_\natural$; in other words N collapses to $K_\natural$ admissibly in the sense of Irwin* (**5**).

Proof. The first part follows from conditions (2) and (3) of well-situatedness by ((**11**) Theorem 2). We obtain the second part by refining Whitehead's proof slightly. His proof goes as follows. Order the simplexes $A_1, \ldots, A_r$ of N that do not meet $K_\natural$ in some order of decreasing dimension; by condition (3), for each i, $K_\natural \cap \operatorname{lk}(A_i, N)$ is a (non-empty) simplex, B_i say. The collapse $N \searrow K_\natural$ is achieved by collapsing $A_i B_i \searrow \dot{A}_i B_i$ in turn, $i = 1, 2, \ldots, r$. Condition (2) ensures that we eventually arrive at $K_\natural$. For the proof to work it is only necessary that the ordering be such that each A_i precedes its faces. Therefore we can re-order so that $A_1, \ldots, A_q$ are in $\mathring{J}$ and $A_{q+1}, \ldots, A_r$ are in $\dot{J}$. Let

$$F = K_\natural \cup \bigcup_{q+1}^{r} A_i B_i.$$

Then $N \searrow F \searrow K_\natural$. The lemma will be proved if we show that $F = N \cap \dot{J} \cup K_\natural$.

If $C \in N \cap \dot{J}$ then C is contained in some $A_i B_i$, $i > q$, and so $N \cap \dot{J} \cup K_\natural \subseteq F$. Conversely, if $i > q$ then $A_i B_i$ has all its vertices in $K \cup \dot{J}$, which is full in J by condition (5), and so $A_i B_i \in K \cup \dot{J}$. But $A_i \notin K$, and so $A_i B_i \in \dot{J}$. Hence $F \subseteq N \cap \dot{J} \cup K_\natural$.

LEMMA 3. *Suppose K, L well situated in J, and let $N = N(K-L, J)$. Then $|N|$ is a regular neighbourhood of $|K|$ mod $|L|$ in $|J|$. If, further, K, L is well situated at the boundary, then $|N|$ meets the boundary regularly.*

Proof. The proof is by induction on $m = \dim J$, and is analogous to the proof of ((**11**) Theorem 22) and ((**12**) Lemma 6). The induction begins trivially with $m = 0$. Assume the lemma for $m-1$, and suppose that J is of dimension m.

First we show that N is a combinatorial m-manifold, i.e. that the link of every vertex x in N is an $(m-1)$-sphere or ball. There are three cases, depending upon whether x lies in $K-L$, $L_\natural$, or $N-K$.

Case (i). $x \in K-L$. Then $\operatorname{lk}(x, N) = \operatorname{lk}(x, J) =$ sphere or ball.

Case (ii). $x \in L_\natural$. Let $J^* = \operatorname{lk}(x, J)$, $K^* = K \cap J^*$, $L^* = L \cap J^*$. It is straightforward to verify that $(K^*)_\natural = K_\natural \cap J^* = \operatorname{lk}(x, K_\natural)$, and hence that K^*, L^* is well situated in J^*. Therefore $\operatorname{lk}(x, N) = N(K^* - L^*, J^*)$, and so, by induction on m,

$$\begin{aligned} |\operatorname{lk}(x, N)| &= \text{a regular neighbourhood of } |K^*| \text{ mod } |L^*| \text{ in } |J^*| \\ &= \text{a regular enlargement of } |(K^*)_\natural| \text{ in } |J^*| \\ &= \text{a ball, by Lemma 1,} \end{aligned}$$

because $(K^*)_\natural = \operatorname{lk}(x, K_\natural)$, which is collapsible by condition (1), since $x \in L_\natural$.

Case (iii). $x \in N-K$. Let J^*, K^*, L^* be as in the last case. Then K^* is a simplex by condition (3), and so K^* is link-collapsible on L^*, because a simplex is link-collapsible on any subcomplex. Also K^* meets $K-L$ because $x \in N$, and so $K^* \neq L^*$. Therefore $(K^*)_\natural = K^*$, and hence $(K^*)_\natural$ is collapsible. We can verify as in the last case that the pair K^*, L^* is well situated in J^*; therefore as before $\operatorname{lk}(x, N) = N(K^* - L^*, J^*)$, and so $\operatorname{lk}(x, N)$ is a ball by induction.

Next we prove that $|N|$ satisfies the second condition for being a regular neighbourhood. $|N|$ is a topological neighbourhood of $|K| - |L|$ in $|J|$, because every point of $|K| - |L|$ is in the open star in J of some vertex in $K-L$, which is an open set of $|J|$ contained in $|N|$. Next we prove that $N \cap L = L_\natural$. For $L_\natural = K_\natural \cap L \subseteq N \cap L$. Conversely, suppose that A is a simplex in $N \cap L$. Since $A \in L$, A does not meet $K-L$. Therefore $Ax \in J$ for some vertex x in $K-L$. Since Ax has all its vertices in $K \cup L$, and $K \cup L$ is full in J by condition (2), we have $Ax \in K \cup L$. But $x \notin L$. Hence $Ax \notin L$, and so $Ax \in K-L$. Therefore $A \in K_\natural$. Therefore $A \in K_\natural \cap L = L_\natural$. Now we shall prove $L_\natural \subseteq \dot{N}$. For given A in $L_\natural$, write $A = xA^*$, where x is some vertex of A. In the notation of case (ii) above, $\operatorname{lk}(A, N) = \operatorname{lk}(A^*, N^*)$, where $N^* = N(K^* - L^*, J^*)$. Now $A^* \in L_\natural \cap J^* = (L^*)_\natural$, and so, by induction on m, we have A^* in the boundary of N^*. Therefore $\operatorname{lk}(A^*, N^*)$ is a ball, and so $A \in \dot{N}$. Hence $L_\natural \subseteq \dot{N}$. Therefore $L_\natural \subseteq \dot{N} \cap L \subseteq N \cap L = L_\natural$. Therefore there is equality $L_\natural = \dot{N} \cap L = N \cap L$, and condition (2) is proved.

The third and last condition for $|N|$ to be a regular neighbourhood is that $N \searrow K_\natural$, which comes from Lemma 2.

For the second part of Lemma 3, we assume, further, that K, L are well situated at the boundary. By what we have already proved, we deduce that if $N_1 = N((K \cap \dot{J}) - (L \cap \dot{J}), \dot{J})$ then $|N_1|$ is a regular neighbourhood of $|K \cap \dot{J}|$ mod $|L \cap \dot{J}|$ in $|\dot{J}|$. The proof of the lemma will be completed if we show that

$$\overline{(N \cap \dot{J}) - L} = N_1.$$

If $A \in N_1$ then $A \in N(x, \dot{J})$ for some x in $(K - L) \cap \dot{J}$, and

$$N(x, \dot{J}) \subseteq N(x, J) \subseteq N.$$

Therefore $N_1 \subseteq N \cap \dot{J}$. Conversely, suppose that $A \in N \cap \dot{J}$. Then $A \in N(x, J)$ for some x in $K - L$. Therefore xA has all its vertices in $K \cup \dot{J}$, which is full in J by condition (5), and so $xA \in K \cup \dot{J}$. Therefore xA lies in either $\dot{J}$ or K. If $xA \in \dot{J}$ then $A \in N_1$. Alternatively, if $xA \in K$ then $A \in K$, and so either $A \in K - L$, whence $A \in (K - L) \cap \dot{J} \subseteq N_1$, or else $A \in L$. We have shown that

$$N_1 \subseteq N \cap \dot{J} \subseteq N_1 \cup L.$$

Therefore $N_1 - L = (N \cap \dot{J}) - L$. But since $|N_1|$ is a regular neighbourhood of $|K \cap \dot{J}|$ mod $|L \cap \dot{J}|$, the interior of N_1 does not meet L. Consequently $N_1 = \overline{N_1 - L} = \overline{(N \cap \dot{J}) - L}$, as desired. The proof of Lemma 3 is complete.

Proof of Theorem 1. We are given a second-derived neighbourhood N of X mod Y in M, which we want to show is regular. N is formed by choosing a triangulation J, K, L of M, X, Y, choosing a second-derived J'' of J, and defining

$$N = |N(X - Y, J'')|.$$

Let J' be the first derived of J. Let J^* be the first derived of J' mod $K' \cup L'$; that is to say we form J^* by starring in some order of decreasing dimension all the simplexes of $J' - (K' \cup L')$, at the same points that were used to form J''. The theorem will then follow from Lemmas 3 and 4 below.

LEMMA 4. (i) $N = |N(X - Y, J^*)|$.

(ii) *If X is link-collapsible on Y then K^*, L^* is well situated in J^*.*

(iii) *If, further, $X \cap \dot{M}$ is link-collapsible on $Y \cap \dot{M}$ then K^*, L^* is well situated at the boundary.*

Proof. First form J^{**} from J^* by starring all the simplexes in $L^* - L_\natural^* = L' - L_\natural'$. This process leaves $N(X - Y, J^*)$ untouched, because if $A \in L' - L_\natural'$ then $A \notin N(X - Y, J^*)$ for the following reason.

Suppose not: then there is a vertex x in $K' - L'$ such that $xA \in J^*$. The simplex xA has all its vertices in $K' \cup L'$, which is full in J', because we have taken first deriveds, and hence is also full in J^*. But $x \notin L'$, and so $xA \in K' - L'$. Therefore $A \in K_\natural'$. Therefore $A \in K_\natural' \cap L' = L_\natural'$, which is a contradiction. We have shown that

(iv) $N(X - Y, J^*) = N(X - Y, J^{**})$.

Now star the simplexes in $K'_{\natural}$, and form J'' from J^{**}. Let B denote a typical simplex in $K'-L'$, and $\hat{B}$ the point at which it is starred. Then

$$\begin{aligned} |N(X-Y, J^{**})| &= \bigcup_B |N(B-\dot{B}, J^{**})| \\ &= \bigcup_B |N(\hat{B}, J'')| \\ &= N. \end{aligned}$$

This combined with (iv) proves (i).

Proof of (ii). Condition (1) of well-situatedness is true by hypothesis. Condition (2) is true in J' and remains true in J^*. Condition (3) is true in J^{**} by ((**11**) Lemma 4), and remains true in J^* by (iv).

Proof of (iii). Since $(J^*)^{\cdot} = (\dot{J})^*$, condition (4) follows from (ii) applied to the boundary. Condition (5) is already true in J' and remains true in J^*. The proof of Lemma 4 and Theorem 1 is complete.

For the proofs of Theorems 2 and 3 we shall need a lemma about isotopies of balls.

LEMMA 5. *Let $P \subseteq Q \subseteq R$ be a nest of three m-balls, whose boundaries all meet in a common $(m-1)$-face F. Then there exist*

(i) *an isotopy in Q moving P onto Q and keeping F fixed, and*

(ii) *an ambient isotopy of R moving P onto Q keeping $\dot{R}$ fixed.*

Proof. The set-up is homeomorphic to a standard set-up in which $P \subseteq Q \subseteq R$ are m-simplexes with a common $(m-1)$-face F, with the vertex p of P opposite F at the barycentre of Q, and the vertex q of Q opposite F at the barycentre of R. Therefore it suffices to prove the lemma for the standard set-up.

In the standard set-up there are obvious isotopies by straight-line paths, but these are not in the category in which we are working, since they are piecewise algebraic rather than piecewise linear (see the footnote to the proof of ((**4**) Lemma 5)). However, we can define piecewise-linear isotopies as follows. In case (i), represent $P \times I$ as a cone with vertex $p \times 1$ and base $P \times 0 \cup F \times I$, and in case (ii) represent $R \times I$ as a cone with vertex $p \times 1$ and base $R \times 0 \cup \dot{R} \times I$. To obtain the isotopy $P \times I \to Q \times I$ in case (i), and the ambient isotopy $R \times I \to R \times I$ in case (ii), map the base of the cone by the identity, map the vertex $p \times 1$ to $q \times 1$, and join linearly.

Kernels

For the proofs of Theorems 2 and 3 we introduce the technical term 'kernel'. The idea is to construct a second-derived neighbourhood inside a given regular neighbourhood, with respect to a triangulation in which the given neighbourhood collapses *simplicially*, as in the proof of ((**11**) Lemma 11).

More precisely, suppose that X is link-collapsible on Y, and let N_1 be a given regular neighbourhood of X mod Y in M. A *kernel* N of N_1 is constructed as follows. Choose a triangulation J, K, L, G of M, X, Y, N such that G collapses simplicially onto $K_\natural$, by ((**11**) Theorem 7). Choose a second derived J'' of J, and define $N = |N(X-Y, J'')|$. If, further, M is bounded, and $X \cap \dot{M}$ is link-collapsible on $Y \cap \dot{M}$, and N_1 meets the boundary regularly, then we impose the additional restriction on N that $\overline{(N \cap \dot{M}) - Y}$ be also a kernel of $\overline{(N_1 \cap \dot{M}) - Y}$; this can be done by subdividing J if necessary (see ((**11**) Theorems 4 and 7)).

Remarks.

(i) The kernel N is a regular neighbourhood of X mod Y in M (by Theorem 1).

(ii) If $X \cap \dot{M}$ is link-collapsible on $Y \cap \dot{M}$, then N meets the boundary regularly (by Theorem 1).

(iii) N is smaller than N_1, because $N_1 \supseteq |N(X-Y, J)|$, which contains a neighbourhood of $N - Y$ in M.

(iv) If N_1, N_2 are two given regular neighbourhoods, we can choose N to be a kernel of both (again by ((**11**) Theorems 4 and 6)).

LEMMA 6. *Suppose X link-collapsible on Y in M. Let N_1 be a regular neighbourhood of X mod Y in M, and let N be a kernel of N_1. Then there exists an increasing sequence of regular neighbourhoods of X mod Y in M,*

$$N = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_{2r} = N_1,$$

such that, for each i,

$$U_i = U_{i-1} \cup B_i,$$
$$F_i = U_{i-1} \cap B_i,$$

where B_i is an m-ball ($m = \dim M$), and F_i an $(m-1)$-ball facing B_i.

Proof. The proof is similar to that of ((**11**) Lemma 11). With the same notation that was used in defining the kernel, we have $G \searrow K_\natural$ simplicially. Let $A_1, A_2, \ldots, A_{2r}$ be the simplexes of $G - K_\natural$, arranged in such order that the simplicial expansion $K_\natural \nearrow G$ is obtained by first adding A_1 and its face A_2, then adding A_3 and its face A_4, and so on. Let $\hat{A}_i$ be the point at which A_i is starred to form J', and let B_i be the m-ball

$$B_i = |N(\hat{A}_i, G'')|.$$

Define, inductively, $U_i = U_{i-1} \cup B_i$, $U_0 = N$. Then the intersection $U_{i-1} \cap B_i = \dot{U}_{i-1} \cap \dot{B}_i$ is an $(m-1)$-ball by the proof of ((**11**) Lemma 11).

It remains to show that each U_i is a regular neighbourhood of X mod Y. By induction U_i is an m-manifold. The second condition for being a regular neighbourhood is satisfied because U_i lies between two other regular neighbourhoods, $N \subseteq U_i \subseteq N_i$. Finally $U_i \searrow X_\natural$ because

$$N_1 = U_{2r} \searrow U_{2r-1} \searrow \ldots \searrow U_0 = N \searrow X_\natural.$$

The proof of Lemma 6 is complete.

Proof of Theorem 2. We are given two regular neighbourhoods N_1, N_2 of X mod Y in M, where X is link-collapsible on Y. Choose a kernel N of N_1 and N_2, and a kernel N_3 of N. We have to prove that there exists an isotopy in M moving N_1 onto N_2 through a continuous family of regular neighbourhoods, keeping N_3 fixed.

Let U_i be as in Lemma 6. For each i, $1 \leqslant i \leqslant 2r$, we shall construct an isotopy

$$f_i : U_{i-1} \times I \to U_i \times I$$

in U_i moving U_{i-1} onto U_i through a continuous family of regular neighbourhoods (of X mod Y in M), and keeping N_3 fixed. The composition

$$N \times I \xrightarrow{f_1} U_1 \times I \xrightarrow{f_2} U_2 \times I \to \ldots \to U_{2r-1} \times I \xrightarrow{f_{2r}} N_1 \times I \xrightarrow{\subseteq} M \times I$$

gives an isotopy in M moving N onto N_1 keeping N_3 fixed. Similarly there is an isotopy moving N onto N_2, and the reverse of the former followed by the latter gives what we want.

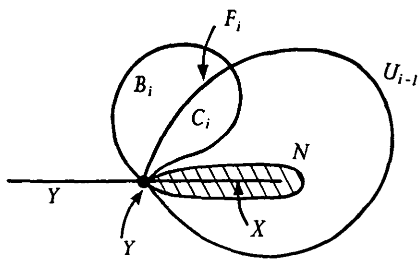

FIG. 5

It remains to construct the isotopy f_i. Suppose therefore that i is fixed, and consider the face F_i of the ball B_i, of Lemma 6. We claim that $\mathring{F}_i$ does not meet N_3. For since N_3 is smaller than N, N contains a neighbourhood of $N_3 - Y$ in M. Therefore $F_i, \subseteq \overline{M-N}$, does not meet $N_3 - Y$. Hence $F_i \cap N_3 = F_i \cap Y$. Now $\mathring{F}_i \subseteq \mathring{N}_1$, and $\mathring{N}_1$ does not meet Y by the second regular-neighbourhood condition. Therefore $\mathring{F}_i \cap N_3 = \mathring{F}_i \cap Y \subseteq \mathring{N}_1 \cap Y = \varnothing$. Consequently $F_i \cap N_3 \subseteq \dot{F}_i$, and so F_i is link-collapsible on $\dot{F}_i \cup N_3$, because a ball is link-collapsible on its boundary.

Let C_i be a regular neighbourhood of F_i mod $\dot{F}_i \cup N_3$ in U_{i-1} (see Fig. 5). Since C_i is a regular enlargement of F_i, it is an m-ball by Lemma 2, meeting B_i in the common face F_i.

By Lemma 5 there is an isotopy in $B_i \cup C_i$ moving C_i onto $B_i \cup C_i$, and keeping $\dot{C}_i - \overset{\circ}{F}_i$ fixed. Extend this by the constant isotopy on $U_{i-1} - C_i$ to the required isotopy in U_i moving U_{i-1} onto U_i, and keeping N_3 fixed. At each stage of this isotopy, the image is a regular neighbourhood, because it is a manifold, lies between two other regular neighbourhoods, and collapses to $X_\natural$ by the image of the collapse $U_{i-1} \searrow X_\natural$. The proof of Theorem 2 is complete.

The proof of Theorem 3 will be a development of that of Theorem 2 in three stages: in Lemma 7 we make the isotopy ambient in the case when M is unbounded; in Lemmas 8 and 9 the bounded case is dealt with; and finally we move the smaller neighbourhood that is to be kept fixed during the isotopy from thesis to hypothesis. Recall the statement of the theorem:

Hypothesis of Theorem 3. *Suppose that X is link-collapsible on Y, and $X \cap \dot{M}$ link-collapsible on $Y \cap \dot{M}$. Let N_1, N_2, N_3 be three regular neighbourhoods of X mod Y in M, meeting the boundary regularly, and such that N_3 is smaller than N_1 and N_2. Let P be the closure of the complement of a second-derived neighbourhood of $N_1 \cup N_2$ mod Y in M.*

Thesis of Theorem 3. *There exists an ambient isotopy of M moving N_1 onto N_2 and keeping $N_3 \cup P$ fixed.*

For the next three lemmas, Lemmas 7, 8, and 9, we assume the hypothesis of Theorem 3, and make the following construction: let N be a kernel of N_1 and N_2, and let N_4 be a kernel of N and N_3.

LEMMA 7. *Suppose M unbounded. Then there exists an ambient isotopy of M moving N onto N_1, keeping $N_4 \cup P$ fixed.*

Proof. The proof is an addendum to the proof of Theorem 2. Using the same notation (with the proviso that for N_3 now read N_4), we show that the isotopy moving U_{i-1} onto U_i can now be realized by an ambient isotopy of M keeping $N_4 \cup P$ fixed. The composition of these ambient isotopies for $i = 1, 2, \ldots, 2r$ will give the required isotopy for Lemma 7.

Continuing with the same notation, let

$$E_i = \mathrm{cl}\,(\dot{U}_i - \dot{U}_{i-1}) = \dot{B}_i - \overset{\circ}{F}_i,$$

which is an $(m-1)$-ball facing the m-ball B_i. We claim that $\overset{\circ}{E}_i$ does not meet P. For $E_i \subseteq N_1$; therefore $E_i \cap Y \subseteq E_i \cap N_1 \cap Y = E_i \cap Y_\natural$, by the second regular-neighbourhood condition, and $E_i \cap Y_\natural \subseteq E_i \cap U_{i-1} = \dot{E}_i$. Hence $\overset{\circ}{E}_i \subseteq N_1 - Y$. But by hypothesis $M - P$ is the interior of a second-derived neighbourhood of $N_1 \cup N_2$ mod Y in M, and therefore $\overset{\circ}{E}_i \subseteq N_1 - Y \subseteq M - P$. We have verified the claim that $\overset{\circ}{E}_i$ does not meet P.

Therefore E_i is link-collapsible on $\dot{E}_i \cup P$, because a ball is link-collapsible on its boundary.

Let D_i be a regular neighbourhood of E_i mod $\dot{E}_i \cup P$ in $\overline{M - U_i}$ (see Fig. 6). Then D_i, being a regular enlargement of the ball E_i, is an m-ball meeting B_i in the common face E_i.

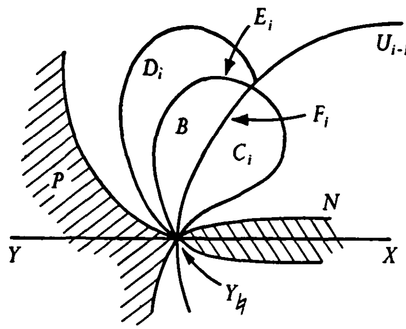

FIG. 6

Let $H_i = B_i \cup C_i \cup D_i$, which is an m-ball whose interior does not meet $N_4 \cup P$, by construction. By Lemma 5 there is an ambient isotopy of H_i moving C_i onto $B_i \cup C_i$, and keeping $\dot{H}_i$ fixed. Extend this by the constant isotopy of $M \backslash H_i$ to the required ambient isotopy of M moving U_{i-1} onto U_i, and keeping $N_4 \cup P$ fixed. The proof of Lemma 7 is complete.

LEMMA 8. *Suppose that M is bounded. Then N_1 collapses admissibly to $X_\natural$, i.e.*

$$N_1 \searrow N_1 \cap \dot{M} \cup X_\natural \searrow X_\natural.$$

Proof. Since N is a kernel of N_1, $N_1 \searrow N$ by Lemma 6. Since N is a second-derived neighbourhood, $N \searrow N \cap \dot{M} \cup X_\natural \searrow X_\natural$ by Lemmas 2 and 4(iii). We shall produce a homeomorphism h of N_1 onto itself, throwing $N \cap \dot{M}$ onto $N_1 \cap \dot{M}$, and keeping $X_\natural$ fixed. Then the image under h of the collapse $N_1 \searrow N \cap \dot{M} \cup X_\natural \searrow X_\natural$ will give what we want.

Let $M^* = \dot{N}_1$, $X^* = X \cap M^*$, $Y^* = Y \cap M^*$, $N_1^* = \overline{(N_1 \cap \dot{M}) - Y}$, and $N^* = \overline{(N \cap \dot{M}) - Y}$. Then, although $X^* \neq X \cap \dot{M}$ and $Y^* \neq Y \cap \dot{M}$ in general, it is nevertheless true that

$$(X^*)_\natural = (X \cap \dot{M})_\natural, \quad (Y^*)_\natural = (Y \cap \dot{M})_\natural,$$

because

$$(X^*)_\natural = \overline{X^* - Y^*} = \overline{(X - Y) \cap \dot{N}_1} = \overline{(X - Y) \cap \dot{M}}$$
$$= \overline{(X \cap \dot{M}) - (Y \cap \dot{M})} = (X \cap \dot{M})_\natural,$$

and

$$(Y^*)_\natural = (X^*)_\natural \cap Y^* = (X \cap \dot{M})_\natural \cap (Y \cap \dot{N}_1)$$
$$= (X \cap \dot{M})_\natural \cap (Y \cap \dot{M}) = (Y \cap \dot{M})_\natural.$$

Therefore by the hypothesis of Theorem 3 and the construction of N, it follows that N_1^* is a regular neighbourhood of X^* mod Y^* in M^*, and N^* is a kernel of N_1^*. Since M^* is closed we can apply Lemma 7 to each component of M^*, and obtain an ambient isotopy f^* of M^* moving N^* onto N_1^*, and keeping X^* fixed. We wish to extend f^* to an ambient isotopy of N_1, at the same time taking care not to move $X_\natural$.

Now f^* is the composition of a finite number of isotopies f_i^*, $i = 1, 2, \ldots, 2r$, where f_i^* is an ambient isotopy of M^* keeping everything fixed except an $(m-1)$-ball H_i^*, whose interior does not meet X^* and hence does not meet $X_\natural$. Therefore H_i^* is link-collapsible on $\dot{H}_i^* \cup X_\natural$. Let H_i be a regular neighbourhood of H_i^* mod $\dot{H}_i^* \cup X_\natural$ in N_1. Then H_i, being a regular enlargement of H_i^*, is an m-ball meeting $\dot{N}_1$ in the face H_i^*. Hence H_i is homeomorphic to a cone on H_i^* (the homeomorphism keeping H_i^* fixed), and so the ambient isotopy $f_i^* \mid H_i^*$ of H_i^* can be extended conewise to an ambient isotopy f_i of H_i keeping $\dot{H}_i - \mathring{H}_i^*$ fixed. Extend f_i to an ambient isotopy of N_1, fixed outside H_i. Let f be the composition of the f_i, $i = 1, \ldots, 2r$. By our construction, f is an ambient isotopy of N_1 moving N^* onto N_1^* (since it is an extension of f^*), and keeping $X_\natural$ fixed.

Now $Y^* = Y_\natural \subseteq X_\natural$, and so f keeps Y^* fixed. Therefore f moves $N \cap \dot{M}$ onto $N_1 \cap \dot{M}$ because $N \cap \dot{M} = N^* \cup Y^*$ and $N_1 \cap \dot{M} = N_1^* \cup Y^*$. Consequently the end of the isotopy f gives the homeomorphism h that we want, throwing $N \cap \dot{M}$ onto $N_1 \cap \dot{M}$ and keeping $X_\natural$ fixed. The proof of Lemma 8 is complete.

LEMMA 9. *Suppose M bounded. Then there exists an ambient isotopy of M moving N onto N_1 and keeping $N_4 \cup P$ fixed.*

Proof. The lemma is the 'bounded' analogue of Lemma 7, and, as in the proof of Lemma 7, we construct for each i an ambient isotopy of M moving U_{i-1} onto U_i, keeping $N_4 \cup P$ fixed. The boundary is a potential source of trouble, because if, for some i, it happened that $B_i \cap \dot{M} = \dot{B}_i - \mathring{F}_i$, then an ambient isotopy would be impossible: we should have to push a bit of M off the boundary, as it were. The trouble is precisely what does occur if a non-admissible collapse $N_1 \searrow X_\natural$ is used in Lemma 6 to construct the sequence of U_i's, and it was to avoid this contingency that we proved the existence of an admissible collapse in Lemma 8.

More precisely, if the admissible collapse of Lemma 8 is used to define the ordering $A_1, A_2, \ldots, A_{2r}$ of the simplexes in the proof of Lemma 6, then there exists a q such that

$$A_i \subseteq \dot{M}, \quad 1 \leqslant i \leqslant q,$$
$$\mathring{A}_i \subseteq \mathring{M}, \quad q < i \leqslant 2r.$$

If $i > q$, then $B_i \subseteq \mathring{M}$, because B_i is the closed star, in a second derived, of an interior vertex in a first derived, of a triangulation of M. Therefore we can proceed to define the ambient isotopy as in Lemma 7. If $i \leqslant q$, then B_i meets $\dot{M}$, and we proceed as follows.

For the rest of the proof of this lemma we use the superscript star to denote intersection with $\dot{M}$, $U_i^* = U_i \cap \dot{M}$, etc. Then B_i^* is an $(m-1)$-ball facing B_i, and meeting F_i in the common $(m-2)$-face F_i^*, and such that

$$U_i^* = U_{i-1}^* \cup B_i^*, \quad F_i^* = U_{i-1}^* \cap B_i^*.$$

Let J_i denote the $(m-2)$-ball $J_i = \dot{F}_i - \mathring{F}_i^*$, with boundary the $(m-3)$-sphere J_i^*.

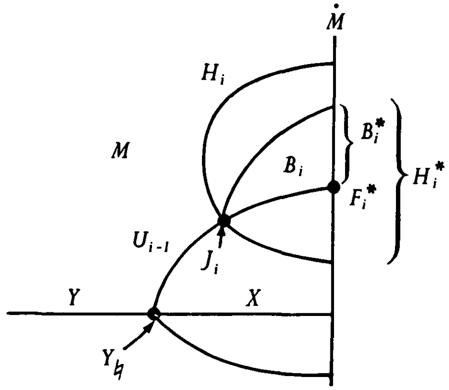

FIG. 7

As in the proofs of Theorem 2 and Lemma 7, we can show that $B_i \cap (N_4 \cup P) \subseteq J_i$, and so B_i is link-collapsible on $J_i \cup N_4 \cup P$, and B_i^* is link-collapsible on $J_i^* \cup N_4^* \cup P^*$. Let H_i be a regular neighbourhood of B_i mod $J_i \cup N_4 \cup P$ in M that meets the boundary regularly (such exist by Theorem 1). Then H_i is an m-ball meeting $\dot{M}$ in the face H_i^*, which is a regular neighbourhood of B_i^* mod $J_i^* \cup N_4^* \cup P^*$ (by definition of meeting the boundary regularly). See Fig. 7. Therefore the pair H_i, B_i is homeomorphic to the cone on the pair H_i^*, B_i^*, the homeomorphism keeping H_i^*, B_i^* fixed. By Lemma 5 there is an ambient isotopy of H_i^*, fixed on the boundary, and moving $U_{i-1} \cap H_i^*$ onto $U_i \cap H_i^*$. We can extend this, first conewise to an ambient isotopy of H_i, and then by the constant isotopy on the complement, to an ambient isotopy of M, moving U_{i-1} onto U_i, and fixed outside H_i. The proof of Lemma 9 is complete.

Proof of Theorem 3. Lemmas 7 and 9, applied to both N_1 and N_2, give an ambient isotopy f of M moving N_1 onto N_2 and keeping $N_4 \cup P$ fixed.

Let N_5 be a kernel of N_4, and let $P' = \overline{M - (N_1 \cap N_2)}$. Now N_4 is a kernel of N_3, and P' is contained in the closure of the complement of a second-derived neighbourhood of N_3 mod Y in M, because N_3 is smaller than

N_1 and N_2. Therefore we can appeal to Lemmas 7 and 9 to obtain an ambient isotopy of M moving N_4 to N_3, and keeping $N_5 \cup P'$ fixed. In particular, the end of this isotopy furnishes a homeomorphism h of M onto itself, throwing N_4 onto N_3, throwing N_1 onto itself and N_2 onto itself, and keeping P fixed. The image of f under h, or, more precisely, the composition

$$M \times I \xleftarrow{h\times 1} M \times I \xrightarrow{f} M \times I \xrightarrow{h\times 1} M \times I,$$

gives the ambient isotopy of M that we want, moving N_1 onto N_2, and keeping $N_3 \cup P$ fixed. The proof of Theorem 3 is complete.

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