PIECEWISE LINEAR TRANSVERSALITY

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We prove transversality theorems for piecewise linear manifolds, maps and polyhedra. Our main result is that given two closed manifolds contained in a third, then one can be ambient isotoped until it is transversal to the other. This result is then extended to maps and polyhedra.

The transversality theory for smooth manifolds was initiated by Thom in his classical paper [8], and has been extended to piecewise linear manifolds by Williamson [9]. For both of these authors the raison d'être was cobordism theory, and the transversality theorem that was needed was the following: given a map $M \to Q$ between manifolds then it can be homotoped transversal to a given submanifold P of Q. In order to prove this P was assumed to have a normal bundle, and the technique was to slide the map locally along the fibres, and then globalise by using Baire's theorem in the function space.

However, as yet the existence of normal bundles in the piecewise linear category is an open question. Haefliger and Wall [3] have shown that normal bundles exist in the stable range, but Hirsch [5] has shown that normal disc bundles do not always exist in the unstable range, and this suggests that normal bundles may not exist either. To cope with this difficulty Rourke and Sanderson [7] have recently introduced block bundles, which differ from ordinary bundles by having a block over each simplex instead of a disc over each point. Similar theories have been introduced by Haefliger [4] and Morlet [6]. The importance of block bundles is that in the piecewise linear category normal block bundles exist and are unique. With this tool Rourke and Sanderson [7, II] have proved a transversality theorem similar to Theorem 1 below. Like ours it is an isotopy theorem, unlike the homotopy theorem of Williamson mentioned above. Like us, they use direct geometric methods rather than function space methods, because in the function space of embeddings, those that are transversal do not form an open set.

When generalising to polyhedra, block bundles are no good because the regular neighbourhood of a polyhedron in a manifold is not a block bundle. The technique that we use deals with both submanifolds and subpolyhedra. The idea is to triangulate the ambient manifold so that one object is a subcomplex, and then ambient isotope the other so that it cuts across the triangulation—we call this *transimplicial*. As a result the two objects will then be transversal.

We now confine ourselves to definitions of transversality and statements of results. For descriptions of the transimplicial tool, and detailed proofs see [1], [2]. We work throughout in the piecewise linear category; for the standard definitions and properties of this category see [10]. All manifolds will be *closed* (compact without boundary), all submanifolds locally flat (which is always the case in codimension $\geqq 3$ by [10]), and all polyhedra compact. We remark that the definition and results of the first section can be extended to admit manifolds with boundary.

Transversality for manifolds. We shall use D^t to denote a (piecewise linear) t-disc, with centre 0.

DEFINITION 1. Let Q^q be a manifold and M^m, P^p submanifolds. Then M and P are *transversal at the point* $z \in M \cap P$ if there is an embedding

$$h\colon D^{m+p-q} \times D^{q-m} \times D^{q-p}, 0 \times 0 \times 0 \to Q, z$$

such that

$$h^{-1}M = D^{m+p-q} \times 0 \times D^{q-p},$$

$$h^{-1}P = D^{m+p-q} \times D^{q-m} \times 0.$$

M and P are *transversal* in Q if they are transversal at each point of their intersection. If M and P are transversal in Q, then $M \cap P$ is a (closed locally flat) submanifold of both M and P of dimension $m+p-q$.

THEOREM 1. *Given M, $P \subset Q$, then M can be ambient isotoped transversal to P by an arbitrarily small ambient isotopy of Q.*

THEOREM 2. *Given manifolds $M \subset P \subset Q$, there exists a fourth manifold N, contained in Q, that intersects P transversally in M.*

(N will have a boundary. However, ∂N and P will not intersect and so the transversality of N and P makes sense under Definition 1.)

Transversality for maps.

DEFINITION 2. Let P, Q, M be manifolds with $P \subset Q$. Given a map $f\colon M \to Q$, let x be a point of M such that $fx \in P$. The map f is *transversal to P at x* if there is a commutative diagram

$$\begin{array}{ccc} D^{q-p} \times D^{m+p-q}, 0 \times 0 & \xrightarrow{1 \times k} & D^{q-p} \times D^{p}, 0 \times 0 \\ \phi \downarrow & & \downarrow \psi \\ M, x & \xrightarrow[f]{} & Q, fx \end{array}$$

where ϕ, ψ are embeddings onto neighbourhoods of x, fx respectively, such that $\psi^{-1}P = 0 \times D^p$, and k is a map $D^{m+p-q} \to D^p$. f is *transversal* to P if it is transversal at all such points x. We see straightway from the definition that if f is transversal to P then $f^{-1}P$ is a (closed locally flat) submanifold of M of codimension $q-p$.

THEOREM 3. *Given $P\subset Q$ and $f\colon M\to Q$, then f can be ambient isotoped transversal to P by an arbitrarily small ambient isotopy of Q.*

Notice that an ambient isotopy of a *map* is the composite of the map with an ambient isotopy of the target manifold.

Transversality for polyhedra. Let X be a polyhedron. We shall associate with each point $x\in X$ an integer $I(X,x)$, called the *intrinsic dimension* of X at x, as follows.

DEFINITION 3. $I(X, x)$ is the largest integer t for which there is a cone V, with vertex v, and an embedding

$$f\colon D^t\times V,\ 0\times v\to X,\ x.$$

REMARK. Equivalent definitions of intrinsic dimension are:

(a) There is a triangulation of X with x in the interior of a t-simplex if and only if $t\leqq I(X, x)$.

(b) Consequently the set of all points of intrinsic dimension $\leqq t$ is the same as the intersection of the t-skeletons of all triangulations of X.

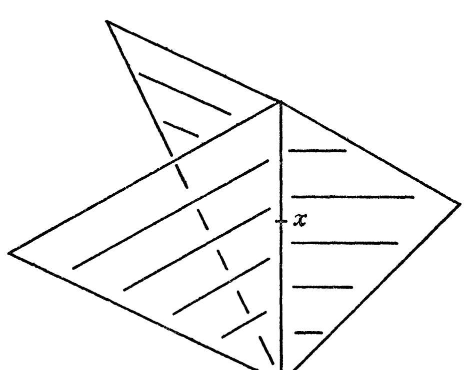
(c) Let L be the link of x in X (defined up to piecewise linear homeomorphism). Then $I(X, x)$ is the largest t such that L is a t-fold suspension.

EXAMPLES.

1. If X is a manifold of dimension n, then

$$\begin{aligned} I(X, x) &= n \text{ if } x \text{ lies in the interior of } X,\\ &= n-1 \text{ if } x \text{ lies in the boundary of } X.\end{aligned}$$

2. Let X, x be as illustrated, then $I(X, x)=1$. Here V is the cone on three points.



3. Let X be the double suspension of a Poincaré sphere. It is a well known conjecture that X is topologically homeomorphic to S^5. But from the point of view of the piecewise linear structure given by the double suspension

$$\begin{aligned} I(X, x) &= 1 \text{ for } x \text{ on the suspension circle,} \\ &= 5 \text{ for other points.} \end{aligned}$$

Suppose now we have two subpolyhedra X, Y of a manifold Q. The *codimension* of $X \subset Q$ is the dimension of Q minus that of X. Let z be a point of $X \cap Y$ and suppose

$$I(X, z) = t,$$
$$I(Y, z) = u.$$

DEFINITION 4. The polyhedra X, Y are *transversal at* z if there is an embedding

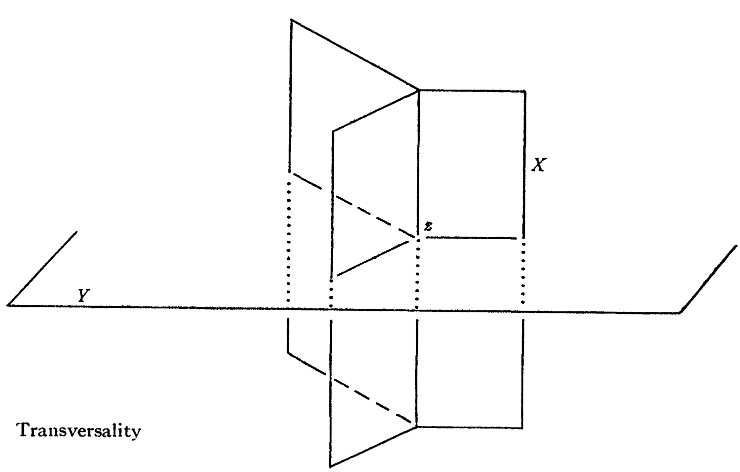
$$h: D^{t+u-q} \times D^{q-t} \times D^{q-u}, 0 \times 0 \times 0 \to Q, z$$

and subcones $V \subset D^{q-t}$, $W \subset D^{q-u}$ (a disc is regarded as a cone with vertex 0 and its boundary as base) such that

$$h^{-1}X = D^{t+u-q} \times V \times D^{q-u},$$
$$h^{-1}Y = D^{t+u-q} \times D^{q-t} \times W.$$

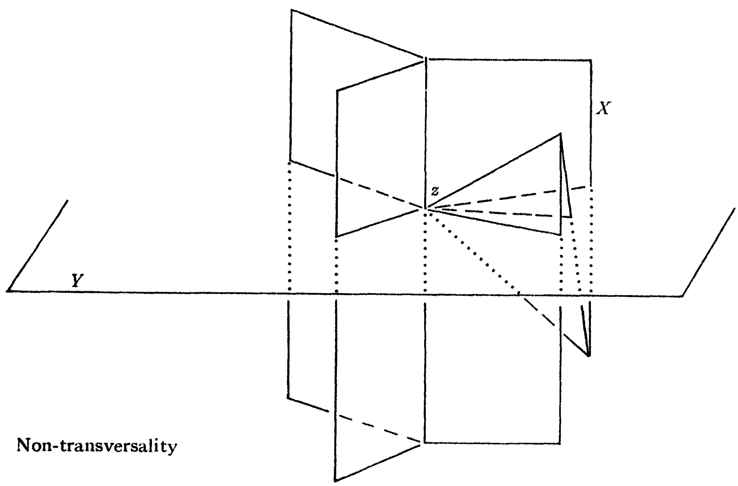
X and Y are *transversal* in Q if they are transversal at each point of their intersection.

In the case where X and Y are closed manifolds, t, u become their respective dimensions, V, W each reduce to a single point, and so the definition agrees with that given earlier. Transversality and non-transversality situations are illustrated below.



Transversality

THEOREM 4. *Given X, $Y \subset Q$ both of codimension $\geqq 3$, then X can be ambient isotoped transversal to Y by an arbitrarily small ambient isotopy of Q.*



Non-transversality

Relative transversality. We have not been able to prove a relative transversality theorem, and the block bundle theory suggests there is an obstruction involved, see [1], [7, II]. The simplest outstanding question is as follows: Given two spheres transversal in the boundary of a ball, do they span transversal discs in the ball?

REFERENCES

1. M. A. Armstrong and E. C. Zeeman, *Transversality for piecewise linear manifolds* (to appear).

2. M. A. Armstrong, *Transversality for polyhedra* (to appear).

3. A. Haefliger and C. T. C. Wall, *Piecewise linear bundles in the stable range*, Topology **4** (1965), 209–214.

4. A. Haefliger, *Knotted spheres and related geometric problems*, Abstracts of reports of I.C.M., Moscow, 1966.

5. M. W. Hirsch, *On tubular neighborhoods of manifolds.* I, II, Proc. Cambridge Philos. Soc. **62** (1966), 177–185.

6. C. Morlet, *Les voisinages tubularies des variétés semi-linéaires*, C. R. Acad. Sci. Paris **262** (1966), 740–743.

7. C. P. Rourke and B. J. Sanderson, *Block bundles.* I, II, III (to appear).

8. R. Thom, *Sur quelques propriétés globales des variétés différentiables*, Comment Math. Helv. **28** (1954), 17–86.

9. R. E. Williamson, *Cobordism of combinatorial manifolds*, Ann. of Math. **83** (1966), 1–33.

10. E. C. Zeeman, *Unknotting combinatorial balls*, Ann. of Math. **78** (1963), 501–526.

11. ———, *Seminar on combinatorial topology*, mimeographed notes, Inst. Hautes Etudes Sci., Paris, 1963.

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