An Essay on Dynamical Systems

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Classification.

Twenty years ago it would have seemed an impossible task to try and classify all ordinary differential equations. There were just too many different types. However the impact of structural stability, and related concepts makes classification a plausible goal (see [14, 17]). The whole subject of differential equations has acquired a new cohesion, and the many special equations of classical literature are now seen to fit as parts of a general overall pattern [3] . The more that the pattern emerges, the clearer it becomes which properties of differential equations are fundamental, and which are really only technical difficulties. To give an example: one only has to compare the complexity of the earlier topological dynamics with the simplicity of the new differentiable dynamics to realise how the former had unwittingly misdirected the subject. The emphasis on topology-without-differentiability had allowed the subject to stray up a side alley, and lose itself in low-dimensional pathology. With the new ideas the generic situation is n-dimensions is now much clearer.

Structural stability.

Let us be more explicit, by describing the compact autonomous case. Let M be a smooth manifold, and for simplicity of exposition suppose M is compact. Let X be a vector field on M. The ordinary differential equation is $\dot{x} = X$, when $x \in M$. By the existence theorem X determines a unique flow or dynamical system φ on M. More precisely a dynamical system is an R-action on M (where R denotes the reals), in other words φ is a smooth map $\varphi : R \times M \to M$, such that $\forall t \in R$, $\varphi_t : M \to M$ is a diffeomorphism, where $\varphi_t(x) = \varphi(t,x)$, and such that $\varphi_s \varphi_t = \varphi_{s+t}$. To define structural stability we must choose an equivalence relation and a topology on the set $\mathfrak{X}$ of all vector fields on M, as follows. Define X, X' to be <u>equivalent</u> if $\exists$ a homeomorphism of M throwing X-orbits onto X'-orbits. Choose the C^1-topology on $\mathfrak{X}$; in other words X,X' are close if both the vectors and their first partial derivatives are close. Then define X to be <u>structurally stable</u> if it has a neighbourhood of equivalents in $\mathfrak{X}$.

We remarked that structural stability is quite different from classical notions like Lyapunov-stability; the latter refers to individual orbits, whereas the former refers to the whole system. Lyapunov-stability says that in a fixed system the orbit does not change much if we perturb the initial conditions. Structural stability says that if we perturb the whole system the global quality is preserved (allowing for arbitrary initial conditions). Thoughout this essay we shall always use the word <u>stable</u> to refer to concepts like structural stability and use the word <u>attractor</u> to refer to individual closed orbits that are Lyapunov-stable. We shall see that in higher dimensions attractors can be more complicated and subtle than individual orbits, and are in a sense the basic indecomposable units.

Structural stability was obviously part Poincare's intuitive thinking in 1890, but was not formally introduced until 1937 by Pontrjagin and Andronov [1]. It became a dominant theme when Peixoto [6] proved two beauitful theorems in 1961 of density and classification. Peixoto's theorems were very limited because they only referred to 2-manifolds, but they were important because they set the guidelines for research during the 1960's, and many workers struggled hard to prove n-dimensional analogues. Therefore it is worthwhile examining them in more detail.

Peixoto's first theorem proved that if dim M = 2 then structurally stable systems are open dense in $\mathfrak{X}$. Openness is trivial but density is deep. Density means that for modelling experiments one can ignore unstable systems, and need only use stable systems. Moreover if the experimental inaccuracy is smaller than the neighbourhood of stability, then the model remains valid in spite of experimental perturbations. Therefore from the point of view of applied mathematics structural stability is an attractive notion. Meanwhile from the point of view of pure mathematics structural stability is equally attractive, because stable systems are vastly simpler that non-stable, and although the classification of all systems remains intractible, the classification of stable systems becomes plausible.

In 1966 hopes were dampened, but research was stimulated, by results of Smale [13] and Williams, showing that for dim M $\geqslant$ 3 structurally stable systems were not dense. Many refinements of the definitions were explored, many properties were proved generic, and many subtle examples were concocted - see the surveys [3, 14]. But to my mind one of the most striking results came in 1972 when Shub [10] (and Hirsch), using a result of Smale [15] showed that structurally stable systems are dense in the C^o-topology on $\mathfrak{X}$. In other words they are open-not-dense in the C^1-topology and dense-not-open in the C^o-topology. This curious but remarkable result retains the attractiveness mentioned above to both pure and applied mathematics, because it is just what is needed for both classification and modelling. For convenience we shall refer to this result as the Density Theorem. The proof uses handlebody decomposition of manifolds, one of the main tools of differential topology. An amusing anecdote is that Smale invented the tool on a beach in Rio in 1960, and then eleven years later as he stumbled across the same beach it suddenly occured to him how to use the tool in this context.

Attractors and basic sets.

We now turn to Peixoto's second theorem. He classified structurally stable systems on a 2-manifold by showing them to be Morse-Smale [3, 11] . Before describing Morse-Smale systems we shall introduce the non-wandering set Ω , because this will enable us to discuss at the same time the n-dimensional classification.

A point $x \in M$ is called wandering if some neighbourhood of it wanders away and never comes back; more precisely $\exists\, N, x \in N \subset M$, and $\exists\, t_o$, such that $\forall t > t_o$, $N \cap \varphi_t N = \emptyset$. Let Ω be the set of non-wandering points. In some sense Ω is the heart of the dynamical system, because it tells us qualitatively about the long term behaviour: Ω contains the attractors, the points and sets towards which almost all orbits eventually flow. Each attractor has its basin of attraction, and from Ω can also be deduced the boundaries of the various basins.

Let us consider some examples. Suppose X is a gradient system, on other words $X = -$ grad V for some generic potential $V: M \to R$. Then Ω is the finite set of fixed points, the sinks, sources and saddles , or in our terminology attractors, repellors and saddle-points. In the definition (see [3, 11]) of a Morse-Smale system, Ω consists of a finite set of fixed points and a finite set of closed orbits; each closed orbit is also an attractor, repellor, or saddle (although in 2-dimensions there are only attractors and repellors because there is not room for a saddle-closed-orbit). Therefore in a Morse-Smale system almost every orbit flows either towards an attractor-point or an attractor-closed-orbit. For example in the Van der Pol equation the manifold $M = R^2$ (non-compact in this case) and Ω consists of one repellor-point and one attractor-closed-orbit. Peixoto's theorem says that this type of behaviour is typical for structurally stable systems in 2-dimensions.

We approach a fundamental question in differential equations theory: what is the typical structure of Ω in n-dimensions? The Density Theorem implies that it is generic property (in the sense of being dense in the C^0-topology) that Ω should have a finite number of connected components. Therefore the generic differential equation on a compact n-manifold only has a finite number of attractors. The components of Ω are called <u>basic sets</u>. A great deal of research has been directed at unravelling the structure of basic sets, and in particular the structure of attractors (see for example [18]). In other words we are trying to answer the question: what is the n-dimensional analogue of an attractor-closed-orbit?

Before 1960 no examples were known, but now there are many revealing examples, and both generic properties and general patterns are emerging. Some basic sets are manifolds, others have Cantor sets as cross-sections, and others can be yet more complicated. Let us confine our attention for the moment to a single attractor in a structurally stable system φ on M, and let us suppose that this attractor A is a compact submanifold of dimension > 1. One of the reasons why such attractors were not discovered earlier was that by a low dimensional fluke dim $A \neq 2$. For, by our assumptions, if φ_A denotes the restriction of the flow to A, then φ_A is itself stable flow on A with non-wandering set $\Omega_A = A$. But by Peixoto's classification theorem above this is impossible if dim A = 2. Hence dim A $\geqslant$ 3.

Anosov flows.

Intuitively one can see why 3-dimensional examples are possible, whereas 2-dimensions are not, by looking at the beautiful examples of Anosov flows, (partly suggested by Thom and proved stable by Anosov [2] in 1962). These flows are characterised by being everywhere hyperbolic, or saddle-like; in other words any small section transverse to the flow can be written as the product of two subsections, one of which expands as it flows along, and the other of which contracts. Since this requires at least one direction of expansion , one of contraction ,both perpendicular to the flow, the manifold must be at least 3-dimensional. Examples of Anosov flows are the geodesic flows on the unit tangent bundles of Riemannian manifolds with negative curvature [2] . Anosov flows and Anosov diffeomorphisms [see 3, 14] have been studies in depth by Anosov, Sinai, Moser, Mather, Bowen, Franks, Walters, Manning and many others, by means of ergodic theory and entropy, Lie groups and homogeneous spaces, differential geometry, algebraic geometry, homotopy groups, homology theory and spectral sequences, Markov chains and shift automorphisms, zeta functions counting the periodic orbits by means of eigenvalues, proving the stability by means of transversality in function spaces. I wrote this list to indicate how central is the field to mainstream mathematics, how rich and varied are the problems that arise, and how differential equations needs to draw upon every other mathematical discipline.

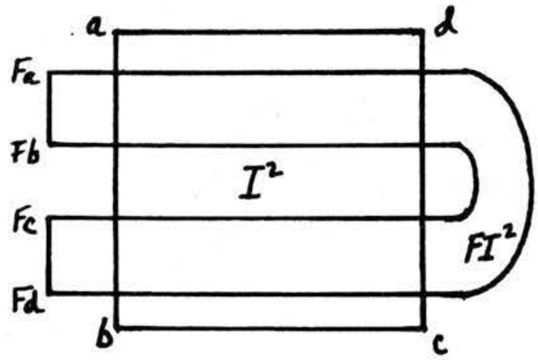
Properties of Basic sets.

I should like to draw attention to two striking properties of basic sets. Firstly in an Anosov attractor there are an infinite number of closed orbits, densely filling the whole manifold. In fact in 1967 Pugh [7] proved that this is generically true for all basic sets. Secondly most orbits are not closed, but spiral densely over the whole basic set. This second property is called transitivity [14], and has been proved for some Anosov flows and for several other examples of basic set, and is conjected to be true in general. In fact the Density Theorem allows us to confine ourselves to transitive basic sets. The first property means that recurrence and resonance must always be present in any non-gradient system while the second property means that the basic sets are indeed the indecomposable units. These two properties must influence our thinking when modelling any complex situation, particularly where there is a lot of feedback, such as in biological systems like the brain.

The horseshoe.

We return to our central theme of analysing basic sets. The most fundamental example of a non-manifold basic set is the Smale horseshoe [12, 14] which he first described in 1961. This is an example that can occur as a saddle inside a 3-dimensional flow, and indeed does occur in perturbations of the forced Van der Pol equation.

I do not have space to describe the horseshoe properly, but let me make a brief attempt. Consider the Cantor set C contained in the unit interval I. The product C^2 is contained in the unit square I^2. We can parametrise C by infinite sequences of 0's and 1's, and further parametrise C^2 by doubly-infinite sequences of 0's and 1's. Let $f: C^2 \to C^2$ denote the shift automorphism which moves each doubly infinite sequence one step to the right. What Smale showed was that it was possible to embed the square I^2 in the sphere S^2, and define a diffeomorphsim $F: S^2 \to S^2$ such that $F|C^2 = f$. The reason he called it the horseshoe was because the image FI^2 of the square was shaped like a horseshoe. Moreover he arranged that the non-wandering set of this diffeomorphism F consisted of a single attractor-point α, a single repellor-point β, and the whole of C^2, which acted like a saddle. (The non-wandering set of a diffeomorphism is analogous to that of a flow). Now <u>suspend</u> F as follows: let M^3 be the manifold obtained from $S^2 \times I$ by glueing the ends together using the diffeomorphism F (actually $M^3 = S^2 \times S^1$ because F is diffeotopic to the identity). Let φ be the flow on M^3 determined by the unit vector field parallel to I.



This is the flow we want to look at. The non-wandering set Ω of φ consists of 3 basic sets, as follows: there is one closed-orbit-attractor, namely $\alpha \times I$ with the ends identified; there is one closed-orbit-repellor, namely $\beta \times I$ with the ends identified; finally there is a saddle-type basic set Γ, namely $C^2 \times I$ with the ends identified by means of the shift automorphism f. Surprisingly Γ is connected as a topological space (although of course C^2 was not connected). The trick is to choose a point $y \in C^2$, whose parameter sequence of 0's and 1's contains as subsequences all possible finite sequences; then, if Z denotes the integers, the f-orbit through y, namely $\{f^n y; m \in Z\}$, is dense in C^2; therefore the φ-orbit of $y \times 0$ spirals densely over Γ. Hence Γ is connected, and so is a basic set. Furthermore Γ is packed densely with an infinite number of closed orbits, because the φ-orbit through $z \times 0$ is closed whenever z is periodic under f. Therefore Γ has the two properties described above. Robbin [8] has shown that the map F is stable, and hence the flow φ is structurally stable.

A classification theorem.

Why is the horseshoe so fundamental? Because with it we can state a generalisation of Peixoto's theorem in n-dimensions. First we can define horseshoes in n-dimensions. These are particular saddle-type basic sets of n-dimensional flows. More precisely an n-horseshoe is the suspension of a subshift of finite type, defined in geometric fashion on a Cantor set in (n-1)-dimensions. Next we can generalise the notion of a Morse-Smale flow as follows. Let us call a flow φ on a compact manifold M^n a Smale flow, if its basic sets are

i) A finite number of fixed points (attractors, repellors & saddles).

ii) A finite number of closed orbits (attractors & repellors only).

iii) A finite number of horseshoes (all saddles).

We should also add a condition that the insets and outsets of basic sets cut transverally. We can now state a generalisation of Peixoto's theorem:

Theorem.

i) Smale flows are structurally stable

ii) Smale flows are dense in the C^0-topology.

The proof of part (i) is due to Robbin [8] and part (ii) is a consequence of the Density Theorem.

It is very satisfactory to have a dense set of models. On the other hand if we apply this theorem to an Anosov flow it seems aesthetically rather a shame to destroy the purity of the latter by approximating it by a Smale system. However Thom points out that this type of "quantization" may be significantly related to thermodynamics.

Diffeomorphisms.

The reader will observe from the horseshoe example how intimately diffeomorphisms are connected with flows. Formally this connection looks even more striking when one observes that a diffeomorphism generates a Z-action, while a flow is an R-action, on a manifold. In fact most of the qualitative phenomena associated with differential equations appear in simpler form, but equal depth, in the study of diffeomorphisms. It is for this reason that Smale's 1967 Survey paper [14] , and much of the succeeding literature has been written in the venue of diffeomorphisms, although the main inspiration and driving force behind the research is the study of differential equations.

Hamiltonian systems.

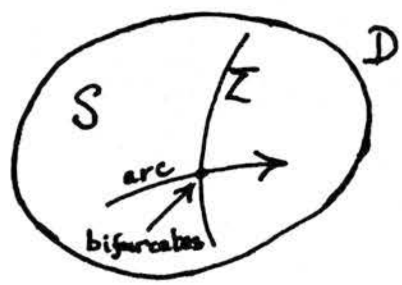
Hamiltonian systems are not structurally stable in the above sense, because they are conservative: energy is conserved and so the orbits lie in the energy levels. If an arbitrarily small damping term is added, then energy is dissipated, and the orbits spiral down to energy minima. The new dissipative flow is not equivalent to the old conservative flow, and hence the structural instability.

However it we confine ourselves to perturbations within the Hamiltonian framework it is possible to develop a theory of Hamiltonian stability, although the latter is still in its infancy. The structure is harder to analyse, because by Liouville's theorem volume is preserved and so there can be no attractors. Therefore there is a greater tendency to use analytic averaging devices, and tools of ergodic theory, rather than the geometric tools of differential topology. Consequently from the qualitative point of view, stable Hamiltonian systems are more complicated than structrually stable systems. Therefore, since physics uses the former and biology the latter, we should expect the forms in physics to be harder to find and more elusive to handle. Indeed in biology geometric forms abound, while in physics averaging devices such as temperature, pressure, density and entropy abound.

However in one area there is a surprising similarity: the ergodic properties of attractors in structurally stable systems have a strong resemblance to ergodic properties of energy levels in Hamiltonian systems. Therefore it is possible that sophisticated ideas of physics may be applicable to complicated states of dynamic homeostasis in biology.

Bifurcation.

Let D denote a space of dynamical systems. For example D might be the space of vector fields on a manifold, or the subspace of gradient fields, or the subspace of Hamiltonian fields, or the space of diffeomorphisms of a manifold. Suppose we had already developed a good theory of stability, and could write $D = S \cup \Sigma$, where S is an open-dense set of stable systems, and Σ the complementary space of unstable systems. So far we have studied the problem of definingand classifying points of S. The next step is to consider parametrised or controlled systems. For example a system that was changing with time would be represented by an arc in D. If the arc crossed Σ , then at that moment the system would change in quality; attractors might bifurcate or basic sets coalesce. The study of such changes is bifurcation theory.



The first task is to analyse the structure of Σ . In general D will be infinite dimensional, but in some cases Σ may be stratified into strata of various codimension. An arc, parametrised by time, would then only meet the strata of codimension 1, and an analysis of such strata would indicate the generic ways a system could bifurcate with time. If 4-dimensional space-time were the control then we should have study strata of codimension $\leqslant 4$ to obtain the generic bifurcations.

Catastrophes.

As yet bifurcation theory is in its infancy, and the only case which has been studied in depth is the bifurcation of gradient systems. This has led to Thom's theory of elementary catastrophes [16, 17, 19]. A gradient system is determined by a potential function $V:M \to R$, and the non-wandering set consists of the singular points of the map V. Therefore not surprisingly the bifurcation of such systems is related to higher dimensional singularities of maps, and using the latter Thom has discovered a remarkable finite classification theorem. He has shown that for a 4-dimensional control space there are only 7 types of bifurcation, governed by 7 particular singularities.

This is a striking and beautiful piece of mathematics for several reasons. Firstly each of the singularities has its own unexpected geometry, which can be described in sufficiently elementary terms as to be accessible to any scientist . Secondly the proof of Thom's theorem due to Mather [4] , Malgrange and Nirenberg [5] uses sophisticated results from several fields induding classical analysis of several variables, both real and complex, function analysis, global analysis, commutative algebra, algebraic geometry, differential topology and differential equations. Therefore it is central to the mainstream of mathematics. It has stimulated considerable growth in the general study of singularities of maps. Thirdly it has applications in many fields of applied mathematics (see for example [16, 17, 19, 20, 21]). It provides a wealth of new models for biology and the social sciences. In particular it provides models for situations where continuous control causes discontinuation jumps of state. For suppose the state of a system lies in some attractor and that this attractor is annihilated by bifurcation - as for example when a minimum coalesces with a maximum and both disappear. Then the state must flow towards a different attractor, and if the speed of flow is large compared with the speed of change of control, then the system will appear to jump into a qualitatively new state - for example the jump in density as slowly increasing temparature causes water to suddenly boil. Previously such discontinuities tended to be handled each by an individual ad hoc model, but bifurcation theory, and in particular catastrophe theory, provides a general method.

Passing from gradient systems to non-gradient systems, the problem of bifurcation becomes much harder, and is virtually unexplored. If the systems involved are Morse-Smale, that is to say the basic sets are just closed orbits, then the elementary catastrophes govern bifurcation. For instance the forced Duffing equation is a beautiful example of the cusp catastrophe. However high-dimensional attractors can bifurcate in a much more complicated way, and already Ruelle and Takens [9] have pointed out that this may give us insight into the mathematics underlying turbulence.

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