THE UMBILIC BRACELET AND THE DOUBLE-CUSP CATASTROPHE

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INTRODUCTION

Our objective is to understand the geometry of the double-cusp catastrophe, or in other words the 8-dimensional unfolding of the germ $f:R^2 \to R$ given by $f = x^4+y^4$. Now 8 dimensions are difficult to visualise and we only partially achieve this objective. So the question arises, why bother with this particular germ? There are several reasons both mathematical and scientific, as follows.

(i) Modality. The double-cusp is the simplest non-simple germ. More precisely any germ in two variables of codimension less than 8 is simple in the sense of Arnold [2,3], but the double-cusp is unimodal. Therefore a study of its geometry will help to give insight into the phenomenon of modality.

(ii) Compactness. The double-cusp is compact, in the sense that the sets $f \leq$ constant are compact. In Arnold's notation [3,4], the double cusp belongs to the family X_9, and in this family there are three real types of germ, according as to whether the germ has 0,2 or 4 real roots. For example representatives of the three types are x^4+y^4, x^4-y^4 and $x^4+y^4 - 6x^2y^2$, respectively, and only the first of these is compact.

Compact germs play an important role in applications [9], because any perturbation of a compact germ has a minimum; therefore if minima represent the stable equilibria of some system, then for each point of the unfolding space there exists a stable state of the system. By contrast, consider the fold-catastrophe x^3, which is not compact; this tends to be an incomplete model of any system, because at the fold point where the equilibrium breaks down, there is a catastrophic jump, but the model does not tell us where the system will jump to. The way to compactify the fold is to add a term x^4; in other words the fold-catastrophe can be regarded as a section of the cusp-catastrophe, which is compact. In this sense we may call the cusp the compactification of the fold. Similarly the double-cusp is important because it is the compactification of each of the three umbilics, the hyperbolic x^3+y^3, the elliptic x^3-3xy^2, and the parabolic x^2y+y^4.

(iii) Coupling. The commonest catastrophe in applications is the cusp, and in some applications two cusps appear, both depending upon the same parameters. In such cases the double-cusp (or one of its non-compact partners) describes the generic way that the two cusps can be coupled together, or can interfere with one another. A study of the geometry is necessary to give a full understanding of such coupling and interference.

(iv) Applications. Samples of the types of application in which the double-cusp appears are as follows. In economics [8] growth and inflation can each be modelled by a cusp, depending upon the same policy parameters such as devaluation, deflation, etc., and the problem is to see how they are coupled, so that one can be cured without harming the other. In linguistics Thom [11,12] uses a compact unfolding of the parabolic umbilic to model basic sentences, and is therefore implicitly using the double-cusp; the four nouns of a basic sentence are represented by the maximal set of 4 minima appearing in the unfolding. In brain-modelling [18] compact germs in 2 variables may be important because the cortex is a 2-dimensional sheet.

In developmental biology if an umbilic appears in the interior of an embryo then, since the embryo continues to exist, the compactification is implicit, and so there should be an accompanying sequence of catastrophes governed by a section of the double-cusp.

In structural engineering [13] the coalescence of two stable post-buckling modes, each governed by a cusp, can generate a highly unstable compound buckling and associated imperfection-sensitivity, governed by a double-cusp. For example this happens in a model due to Augusti* [5,13 Figure 100], consisting of a loaded vertical strut supported at its pinned end by two rotational springs at right-angles, when the strengths of the springs is allowed to coincide. Here the double-cusp is the non-compact $x^4+y^4-6x^2y^2$, with the boundary of stable equilibria representing the failure locus.

CONTENTS

The paper is divided into three sections :

1. The umbilic bracelet.
2. Catastrophe theory.
3. The double-cusp.

In Section 1 we describe the geometry of the discriminant of the real cubic. In Section 2 we establish a new form for the catastrophe map associated with a germ, and show how its singularities refine the canonical stratification of a jet space, which is independent of the unfolding. The new form yields new equations for the cuspoids and umbilics, which help to give further insight into the relationship between their geometries. In Section 3 we apply the results of the two previous sections to explore the geometry of the double-cusp. Other mathematical references containing information about the double-cusp are [1,7,10,17].

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*I am indebted to Michael Thompson for drawing my attention to Augusti's example, and to Tim Poston for pointing out that it was a double-cusp.

SECTION 1 : THE UMBILIC BRACELET.

Since the double-cusp is a quartic form, its unfolding involves the umbilics, namely the cubic forms. Therefore we begin by studying the stratification of the space R^4 of real cubic forms in 2 variables. The point $(a,b,c,d) \in R^4$ corresponds to the form

$$f = ax^3 + bx^2y + cxy^2 + dy^3.$$

The stratification is given by general linear actions as follows. Let $G = GL(2,R)$ be the general linear group of real invertible 2×2 matrices. The left-action of G on the variables by matrix multiplication induces a right-action of G on R^4, as follows : given $f \in R^4$, $g \in G$, define fg by

$$(fg)v = f(gv), \text{ where } v = \binom{x}{y}.$$

Define the stratum containing f to be the G-orbit, fG. The following lemma is classical.

Lemma 1. There are 5 strata in R^4, characterised by the type of roots.

Stratum	Dim	Example	Type of roots.
H, hyperbolic umbilics	4	$x^3 + y^3$	2 complex, 1 real
E, elliptic umbilics	4	$x^3 - 3xy^3$	3 real distinct
P, parabolic umbilics	3	x^2y	3 real, 2 equal
X, exceptional	2	x^3	3 real equal
O, the origin	0	0	indeterminate.

Proof. Real linear action preserves the type of roots. Conversely if f,f' have roots of the same type then there is a real projective map sending roots of f into f', and hence $g \in G$ such that $f' = fg$.

Remark. We call X the exceptional stratum because it underlies the exceptional singularities E_6, E_7, E_8 in Arnold's notation [2]. See also Lemma 13 below.

Discriminant. Define the discriminant $D = P \cup X \cup 0$, the union of the non-open strata. The equation of D is given by eliminating x,y from $f = \partial f/\partial x = 0$:

$$4(ac^3 + b^3 d) + 27a^2 d^2 - b^2 c^2 - 18abcd = 0.$$

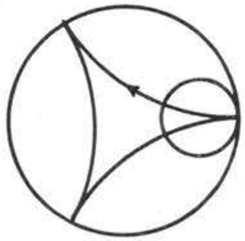
However this is not much help in understanding the geometry, and so we shall pursue a different tack.

Lemma 2. The stratification of R^4 is conical with vertex 0.

Proof. If g = scalar multiplication by λ, then $fg = \lambda^3 f$. Hence the ray through f is contained in the stratum fG.

Remark. The importance of Lemma 2 is that to describe the stratification of R^4 it suffices to describe the induced stratification on the unit sphere $S^3 \subset R^4$, and then take the cone on the latter. We could, further, identify S^3 antipodally and describe the induced stratification of projective space, but we do not do this for two reasons. Firstly, when we come to apply the results to catastrophe theory, antipodal identification confuses maxima and minima, which are important to distinguish. Secondly our immediate aim is to visualise the stratification, and although the projective language is attractive (see Lemma 16), it is slanted towards the algebraic rather than the topological point of view, and consequently can hide some of the geometry. Therefore we shall consider the stratification of S^3, and visualise it in R^3 by removing a point "at infinity."

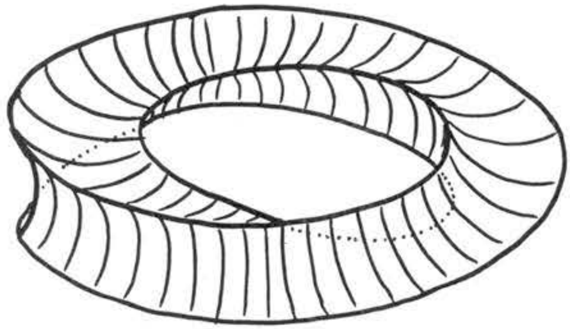
Figure 1.



Recall that a triangular hypocycloid is the locus of a point on a circle of radius 1 rolling inside a circle of radius 3 (see Figure 1). It has 3 cusps and 3 concave sides.

Theorem 1. The induced stratification of S^3 is in the shape of a bracelet*, with a triangular hypocycloid section that rotates ⅓ twist going once round the bracelet. The strata H, E, P, X meet S^3 in the outside, inside, surface, and cusped edge of the bracelet, respectively.

Figure 2.



Remark 1. Since the discriminant is classical, this picture was probably known in the last century, but I have not found a reference to it. In Figure 2 we have sketched the bracelet in R^3, assuming that the point at infinity which has been removed from S^3 was a hyperbolic point. If an elliptic point had been removed the cusped edge would point inwards rather than outwards. The simplest way to project S^3 minus a point onto R^3 is by stereographic projection, but this is geometrically very distorting and in particular badly distorts the hypocycloidal sections. Hence in Figure 2 we

*The name "bracelet" arose when explaining the shape to my wife, who is a jeweller. Subsequently Tim Poston carved beautiful wooden bracelets of this shape.

have drawn a differentiably equivalent image, that preserves the concave curvature of the sections. The geometry is clarified by Lemma 6 below.

Remark 2. In his book [11, p99] Thom suggests that elliptic states are more fragile than hyperbolic (and deduces that males are more fragile than females), because elliptic states are limited and always followed by hyperbolic breaking. His arguement depends upon the stratification of real quadratic forms in 2 variables (see Lemma 10), and the observation that in the real projective plane elliptic forms correspond to the interior of a conic, and hence contain no projective lines. However the applications refer to the umbilics, in other words to cubic forms rather than quadratic forms; and we show in Lemma 5 that there are circles (corresponding to projective lines) both inside and outside the bracelet. Therefore from the qualitative point of view elliptic states are as robust as hyperbolic states, and any comparison between their fragilities would have to be quantitative depending upon some measure of the strata.

The circle group. To prove Theorem 1 it is convenient to use the circle group (which is the maximal torus of G) namely $S^1 = SO(2) = \{g_\theta;\ 0 \leq \theta < 2\pi\}$, where

$$g_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} .$$

In turn, the circle group suggests the convenience of a complex variable $z = x + iy$, because then $g_\theta(z) = e^{i\theta}z$.

Lemma 3. With complex coefficients $(\alpha, \beta) \in C^2$, the generic real cubic form can be written

$$f = \mathcal{R}(\alpha z^3 + \beta z^2 \bar{z}).$$

Proof. Writing $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, then

$$f = \alpha_1(x^3 - 3xy^2) + \alpha_2(-3x^2y + y^3) + \beta_1(x^3 + xy^2) + \beta_2(-x^2y - y^3)$$
$$= (\alpha_1 + \beta_1)x^3 + (-3\alpha_2 - \beta_2)x^2y + (-3\alpha_1 + \beta_1)xy^2 + (\alpha_2 - \beta_2)y^3,$$

which is a permissible change of coordinates for R^4 from (a,b,c,d) because the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -3 & 0 & -1 \\ -3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

is non-singular.

Notation.

A denotes the α-plane, given by $\beta = 0$.

B denotes the β-plane, given by $\alpha = 0$.

A_0 denotes the unit circle in A, given by $|\alpha| = 1$, $\beta = 0$.

B_0 denotes the unit circle in B, given by $\alpha = 0$, $|\beta| = 1$.

We may write $R^4 = C^2 = A \times B$.

Lemma 4. S^1 acts orthogonally on $A \times B$ by rotating A thrice and B once.

Proof. $(fg_\theta)z = f(g_\theta z) = \mathcal{R}(\alpha e^{3i\theta}z^3 + \beta e^{i\theta}z^2\bar{z})$. Therefore $(\alpha,\beta)g_\theta = (\alpha e^{3i\theta}, \beta e^{i\theta})$.

Lemma 5. $A_0 \subset E$, $B_0 \subset H$.

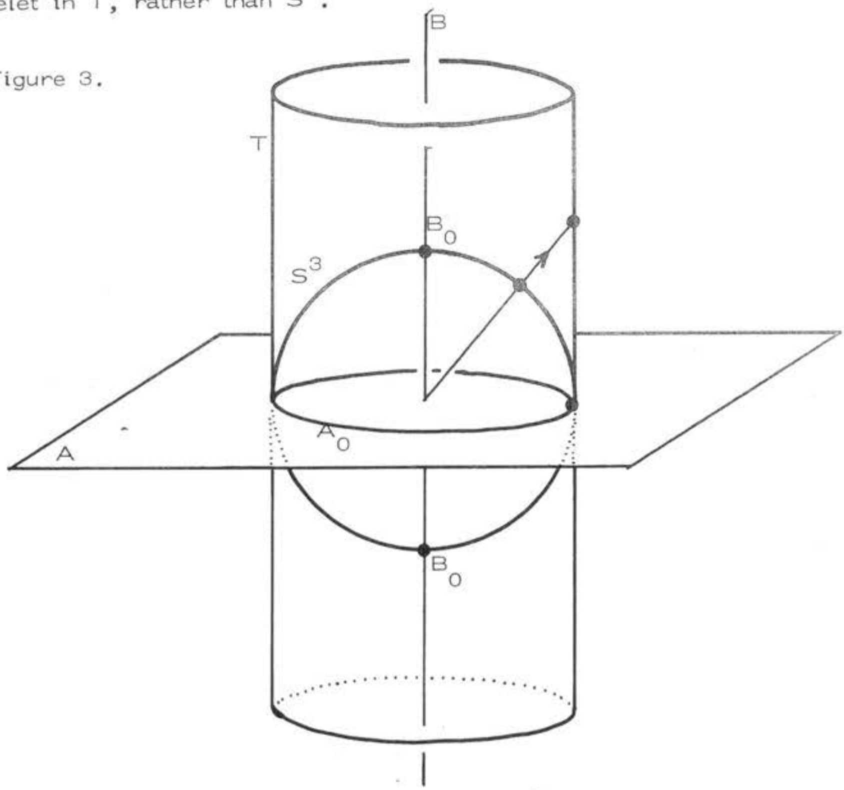
Proof. By Lemma 4 A_0, B_0 are S^1-orbits, and therefore contained in

G-orbits. A_0 contains the point $(\alpha, \beta) = (1,0)$, which corresponds to the form $x^3 - 3xy^2$, which is in E, and therefore $A_0 \subset E$. Similary B_0 contains (0,1), corresponding to $x^3 + xy^2$ in H.

In Figure 2 A_0 is the horizontal core of the bracelet, and B_0 is the vertical axis of the bracelet (together with the point at infinity). Therefore A_0, B_0 represent projective lines in E, H confirming Remark 2 above.

Let $T = A_0 \times B$, the solid torus given by $|\alpha| = 1$. Radial projection from the origin gives a diffeomorphism $S^3 - B_0 \rightarrow T$. This is illustrated in Figure 3, where B is drawn symbolically as 1-dimensional rather than 2-dimensional, and B_0 as a point-pair rather than a circle. By Lemma 5 the bracelet does not meet B_0, and so is projected diffeomorphically into T. Therefore to prove the theorem it suffices to prove the existence of the bracelet in T, rather than S^3.

Figure 3.



Lemma 6. D meets the plane $\alpha = 1$ in the triangular hypocycloid $\beta = 2e^{i\varphi} + e^{-2i\varphi}$, $0 \le \varphi < 2\pi$ (see Figure 1).

Proof. If $(1,\beta) \in D$, then $f = \mathcal{R}(z^3 + \beta z^2 z)$ has a double root in $x : y$. Putting $x = e^{i\theta}$, then

$$f = \tfrac{1}{2}(e^{3i\theta} + e^{-3i\theta} + \beta e^{i\theta} + \beta e^{-i\theta})$$

has a double root in θ. Multiplying by $e^{i\theta}$,

$$\tfrac{1}{2}(e^{4i\theta} + e^{-2i\theta} + \beta e^{2i\theta} + \beta)$$

also has a double root in θ. Therefore the derivative vanishes,

$$2ie^{4i\theta} - ie^{-2i\theta} + i\beta e^{2i\theta} = 0.$$

$$\beta = -2e^{2i\theta} + e^{-4i\theta} \quad .$$

Putting $2\theta = \varphi - \pi$ gives the required formula. Geometrically the formula represents the locus of a point on a circle of radius 1 rolling inside a circle of radius 3, namely the hypocycloid.

Lemma 7. X meets $\alpha = 1$ in the 3 cusp points.

Proof. With a triple root, the second derivative also vanishes. Therefore

$$\frac{d\beta}{d\varphi} = 2ie^{i\varphi} - 2ie^{-2i\varphi} = 0.$$

$$e^{3i\varphi} = 0 .$$

$$e^{i\varphi} = 1, w, w^2, \text{ the cube roots of } 1.$$

$$\beta = 3, 3w, 3w^2, \text{ the three cusp points.}$$

Proof of Theorem 1. Apply the first third of the circle group S^1, for values $0 \le \theta \le \frac{2\pi}{3}$. By Lemma 4 this rotates the circle A_0 once, and gives the plane B a $\tfrac{1}{3}$-twist. Therefore it isotops the plane $\alpha = 1$ once round the torus T, and back onto itself with a ⅓-twist. Therefore it isotops the hypocycloid once round and back onto itself with a ⅓-twist, to form the bracelet shown in Figure 2. Hence the strata P and X meet T in the surface and cusped edge of the bracelet. Meanwhile the strata H and E meet in T in the exterior and interior by Lemma 5. This completes the proof of Theorem 1.

Corollary. The stratification of R^4 can be written parametrically as follows. Let

$$(\alpha,\beta) = \lambda(e^{3i\theta}, \mu e^{i\theta}(2e^{i\varphi} + e^{-2i\varphi})),\ \lambda,\mu \geq 0,\ 0 \leq \theta,\ \varphi < 2\pi .$$

Then the strata are given by

$H : \lambda > 0,\ \mu > 1$.

$E : \lambda > 0,\ 0 \leq \mu < 1$.

$P : \lambda > 0,\ \mu = 1,\ \varphi \neq 0$.

$X : \lambda > 0,\ \mu = 1,\ \varphi = 0$.

$O : \lambda = 0$.

Remark 3. Let G_+ denote the subgroup of G of index 2 consisting of matrices with positive determinant (G is the identity component). Then G_+-orbits in R^4 equal the G-orbits. In particular G_+ acts freely on H, and with index 3 on E. This is related to the geometric fact that there is only one tangent to the hypocycloid from an exterior point, but 3 from an interior point. It also underlies some of the qualitative differences between hyperbolic and elliptic umbilics, for example the bifurcation set of the former has one cusped edge, and the latter three (see Figures 6, 7, 8, 10 and Lemmas 12, 17).

SECTION 2 : CATASTROPHE THEORY.

We recall the construction of the catastrophe map associated with a determinate germ of a function [11,16]. Let $\mathcal{E}$ be the ring of germs of C^∞-functions $R^n \to R$, and m the maximal ideal. Given $f \in \mathcal{E}$, define the Jacobian ideal J of f by $J = (f_1, \ldots, f_n)_{\mathcal{E}}$, where $f_i = \partial f/\partial x_i$ and $(x_1, \ldots, x_n)$ are coordinates for R^n. Note that J is independent of choice of coordinates. Call f determinate if $J \supset M^q$, some q.

Let f be a fixed determinate germ, and suppose $f \in m^k$, $k \geq 3$. We shall mostly assume $f \notin m^{k+1}$, but this is not necessary. Define the *unfolding space of f to be m/J, and define the codimension of f to be dim(m/J). Choose a right inverse $\varepsilon : m/J \to m$ of the projection $m \to m/J$. Define the *unfolding F of f associated with ε to be the map germ

$$F : R^n \times m/J \to R, \text{ given by}$$

$$F(x,c) = fx + (\varepsilon c)x, \quad x \in R^n,\ c \in m/J .$$

Note that the unfolding is not unique, since it depends upon the choice of ε, but is uniquely determined by ε in a coordinate-free way. Define the catastrophe manifold $M \subset R^n \times m/J$ by the equations $F_1 = \ldots = F_n = 0$. Note that the determinacy of f ensures that these equations are independent, and so M is a manifold (or more precisely the germ of a manifold) of the same dimension as m/J. Define the catastrophe map

$$\chi_f : M \to m/J$$

to be the map germ induced by the projection $R^n \times m/J \to m/J$. Let sing χ_f denote the set of singularities of χ_f, and define the bifurcation set to be

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* By Mather's theory [16] our definition of unfolding is universal, but not minimal if $f \notin J$. However the particular germs that we shall be considering here will be homogeneous or quasi-homogeneous, in which case $f \in J$, and so our unfoldings are both universal and minimal.

χ_f(sing χ_f). Let strat χ_f denote the stratification of M induced by sing χ_f. We give a precise definition of strat χ_f below in terms of orbits. For the moment observe that strat χ_f is simpler than the bifurcation set, because the former does not contain self-intersections whereas the latter does (see Figure 6 for example).

For applications it is important to understand the geometry of χ_f, and in particular the geometry of the bifurcation set. This is what we should like to know in the case of the double-cusp, but as yet this problem is unsolved. The problem is made additionally awkward by the non-uniqueness of the bifurcation set, since it depends upon the choice of ε, and is unique only up to diffeomorphism. Since this problem is unsolved, we tackle here the simpler problem of studying sing χ_f and strat χ_f. Here the geometry is made slightly awkward by the fact that M is a non-linear manifold.

Now in applications the non-linearity of M is important, because M frequently represents a graph between cause and effect, and the very essence of catastrophe theory is the multivaluedness of this graph over the unfolding space, together with the catastrophic jumps that occur parallel to R^n, from fold points of M into other sheets of M. However, if we are to try and get an initial grip upon the geometry of strat χ_f, it is useful to replace M by a linear manifold. This is one of the purposes of Theorem 2. The theorem also shows that strat χ_f is a substratification of a canonical stratification, which is, unlike the bifurcation set, independent of choice of unfolding. Surprisingly the canonical stratification is even independent of f, and depends only upon the pair of integers n,k, as follows.

Definition of canonical stratification. Let $\mathcal{G}$ be the group of germs at 0 of C^∞-diffeomorphisms $R^n,0 \to R^n,0$. Then $\mathcal{G}$ acts on the right of $\mathcal{E}$

by composition, leaving m invariant, and hence induces actions upon the powers of the maximal ideal m^k, and the jet spaces m^j/m^k, for $j < k$. Define the canonical stratifications N^k of m^k and $N^{j,k}$ of m^j/m^k to be the sets of $\mathcal{G}$-orbits.

Definition of strat χ_f. Define a map germ $\varphi: R^n \times m/J \to m$ by

$$\varphi(x,c)\xi = F(x+\xi,c) - F(x,c)$$

where $x, \xi \in R^n$, $c \in m/J$ and F is the unfolding of f. Let $\varphi^{-1}N^1$ denote the pull-back under φ of the canonical stratification N^1 of m. Define

$$\text{strat } \chi_f = M \cap \varphi^{-1}N^1 .$$

Note that although N^1 is a global stratification of m, φ is only a map-germ, and so strat χ_f is only a stratification-germ of the manifold-germ M. We verify that this is a reasonable definition by the following lemma.

Lemma 8.

(i) $M = \varphi^{-1}m^2$

(ii) Strat $\chi_f = \varphi^{-1}N^2$

(iii) Sing χ_f is given by the vanishing of the Hessian of F.

(iv) Singularities in the same stratum are equivalent.

Proof. By Taylor expansion

$$\varphi(x,c)\xi = \xi F'(x,c) + \tfrac{1}{2}\xi^2 F''(x,c) + \ldots$$

where primes denote (in tensor notation) the derivatives with respect to x. Therefore

$$\begin{aligned} \varphi(x,c) \in m^2 &\iff \text{coefficient } F' \text{ of } \xi \text{ vanishes} \\ &\iff F_1 = \ldots = F_n = 0 \\ &\iff (x,c) \in M. \end{aligned}$$

Hence $M = \varphi^{-1}m^2$, and strat χ_f is the pull-back of the canonical stratification N^2 of m^2. The Hessian H of F is given by $H = \det F'' = |F_{ij}|$. Let M_i be given by $F_i = 0$. Then $M = M_1 \cap \ldots \cap M_n$. The normal to M_i is

$$(F_{i1},\ldots,F_{in},\frac{\partial F_i}{\partial c_1},\ldots,\frac{\partial F_i}{\partial c_p})$$

where $(c_1,\ldots,c_p)$ are coordinates for m/J. Therefore

$$\begin{aligned}(x,c) \in \text{sing}\ \chi_f &\Leftrightarrow \exists \text{ a tangent of } M \text{ killed by } T\chi_f\\ &\Leftrightarrow \exists v \neq 0,\ v \in R^n, \text{ such that } \binom{v}{0} \subset TM\\ &\Leftrightarrow \exists v \neq 0,\ \forall i,\ \binom{v}{0} \subset TM_i\\ &\Leftrightarrow \exists v \neq 0,\ \forall_i,\ \binom{v}{0} \perp \text{ normal } M_i\\ &\Leftrightarrow \exists v \neq 0,\ F''v = 0\\ &\Leftrightarrow H = 0.\end{aligned}$$

Finally suppose $(x,c) \in \text{sing}\ \chi_f$. Then $\varphi(x,c)$ is the local germ of F at (x,c) in the R^n direction, and by Mather's theory [16, Chapter 7] the stratum of this germ determines the equivalence class of the singularity of χ_f at (x,c). This completes the proof of Lemma 8.

Remark. Note that the converse to (iv) is not true : equivalent singularities do not necessarily lie in the same stratum. For example generic maxima and minima of F lie in distinct open strata of N^2, and hence pull back into distinct open strata of M, although, as regular points of χ_f, they are trivially equivalent. It is important for applications to keep maxima and minima distinct.

We are now ready to state the theorem. Recall our original assumption $f \in M^k$, $k \geq 3$. Therefore $J \subset m^{k-1}$, and $mJ \subset m^k \subset m^2$. Let π denote the projection

$$\pi: m^2/mJ \to m^2/m^k ,$$

let $N = N^{2,k}$ denote the canonical stratification of m^2/m^k, and let $\pi^{-1}N$ denote the pull-back of N under π.

Theorem 2. *The catastrophe map χ_f is equivalent to a map*

$$\chi : m^2/mJ \to m/J$$

such that strat χ refines $\pi^{-1}N$.

Here *refines* means that strat χ is a substratification of $\pi^{-1}N$, in other words each stratum of strat χ is contained in a stratum of $\pi^{-1}N$. Note that N is independent of f, but the refinement in general depends upon both f and the unfolding F (although up to diffeomorphism it is independent of F). The simplest example of refinement can be seen below in the case of the hyperbolic and elliptic umbilics : here $\pi^{-1}N$ is given by a cone in R^3 (see Lemma 10), and the refinements are given by adding respectively one or three generators of the cone (see Examples 3 and 4).

Proof of theorem. Let θ denote the composition

$$R^n \times m/J \xrightarrow{\varphi} m \xrightarrow{\pi_1} m/mJ, \qquad \theta = \pi_1 \circ \varphi$$

where π_1 denotes projection. We shall show that θ is a diffeomorphism germ by proving the derivative $T\theta$ is an isomorphism, as follows. From the definition of φ and F,

$$\begin{aligned}\varphi(x,0)\xi &= F(x+\xi,0) - F(x,0)\\ &= f(x+\xi) - fx\\ &= \xi f'x + \tfrac{1}{2}\xi^2 f''x + \dots\end{aligned}$$

in Taylor expansion. Therefore $T\varphi$ maps $R^n \times 0$ onto the subspace of m spanned by $f_1,\dots,f_n$. Now the determinacy of f ensures that $f_1,\dots,f_n$ are linearly independent modulo mJ, and, furthermore, span J/mJ [13, Lemma 3.8].

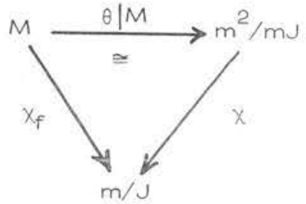
Therefore $T\theta$ maps $R^n \times 0$ isomorphically onto J/mJ. Meanwhile

$$\varphi(0,c) = f + \varepsilon c.$$

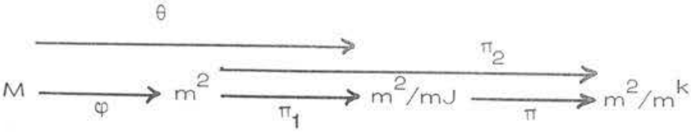
Therefore

$$T\varphi(0 \times m/J) = T\varepsilon(m/J).$$

Now $T\varepsilon$ maps m/J isomorphically onto a complement of J in m, because ε is a right inverse of the projection $m \to m/J$. Therefore $T\varphi$ maps $0 \times m/J$ isomorphically onto this same complement, and hence $T\theta$ maps $0 \times m/J$ isomorphically onto a complement of J/mJ in m/mJ. We have shown that $T\theta$ maps $R^n \times 0$, $0 \times m/J$ isomorphically onto complementary subspaces of m/mJ. Hence $T\theta$ is an isomorphism. Hence θ is a diffeomorphism germ. By Lemma 8, $M = \varphi^{-1}m^2 = \theta^{-1}(m^2/mJ)$. Therefore $\theta\,|M{:}M \to m^2/mJ$ is a diffeomorphism germ, and the map required by the theorem is given by composition $\chi = \chi_f(\theta\,|M)^{-1}$:



Let $\pi_2 = \pi\pi_1$. Then we have compositions



By Lemma 8,

$$\text{strat } \chi_f = \varphi^{-1}N^2 = \varphi^{-1}(\varphi M \cap N^2).$$

Therefore

$$\text{strat } \chi = \theta(\text{strat } \chi_f) = \theta\varphi^{-1}(\varphi M \cap N^2) = \pi_1(\varphi M \cap N^2),$$

because $\pi_1\,|\varphi M$ is a diffeomorphism germ. Now π_2 commutes with the action of $\mathcal{G}$, and so N^2 refines $\pi_2^{-1}N$. Therefore $\varphi M \cap N^2$ refines $\pi_2^{-1}N$. Therefore

$\pi_1(\varphi M \cap N^2)$ refines $\pi_1\pi_2^{-1}N = \pi^{-1}N$. This completes the proof of the theorem.

Indeterminate strata. Call a stratum of N determinate if the germs in the jets of that stratum are determinate, and call it indeterminate otherwise. Let N^* denote the subspace of indeterminate strata. Then N^* has codimension $k - 2$ in m^2/m^k. In the theorem the refinement is limited to $\pi^{-1}N^*$, and so

$$\text{strat } \chi = \pi^{-1}N, \text{ modulo codim } k - 2 .$$

Consequently there is no refinement with the cuspoids ($n = 1$), but there is with the umbilics ($n = 2$) - see below.

Definition. Let $\sigma = \dim(m^k/mJ)$. If $\sigma = 0$ then $\pi = 1$, and $\pi^{-1}N = N$, as in the case of the cuspoids and elliptic and hyperbolic umbilics. If $\sigma > 0$ then $\pi^{-1}N \cong N \times R^\sigma$. There are two possible reasons for $\sigma > 0$:

(i) f not homogeneous; for example $\sigma = 1$ for the parabolic umbilic $x^2y + y^4$, because it is not 3-determinate.

(ii) f has modality > 0 (see [3,4]); for example $\sigma = 1$ for double-cusp $x^4 + y^4$, because $x^2y^2 \notin mJ$.

Lemma 9. $\text{Codim } f = \dfrac{(n+k-1)!}{n!(k-1)!} - (n+1) + \sigma$.

Proof. $\text{Codim } f = \dim(m^2/mJ)$, by determinacy of f,

$$= \dim(\mathcal{E}/m^k) - \dim(\mathcal{E}/m^2) + \dim(m^k/mJ),$$

because $\mathcal{E} \supset m^2 \supset m^k \supset mJ$, which gives the required formula by counting monomials.

Example 1. The cusp, A_3.

Since this is a familiar example, we give the formulae in detail, in order to illustrate the theorem. The cusp catastrophe has symbol A_3 in Arnold's notation [2] and germ x^4, or the equivalent; for convenience of computation we choose the germ $f = \frac{1}{4}x^4$. Here $n = 1$, $k = 4$, $J = m^3$, and therefore $mJ = m^4$, $\sigma = 0$, codim $f = 2$. Choose for the unfolding space m/J the base $\{\frac{1}{2}x^2, x\}$ (regarded as 2-jets) and coordinates (u,v). Therefore a point $c \in m/J$ can be written

$$c = (u,v) = \tfrac{1}{2}ux^2 + vx.$$

Choose $\varepsilon : m/J \to m$ by reinterpreting x^2, x as germs rather than 2-jets. Then the unfolding $F : R \times m/J \to R$ is given by

$$F(x;u,v) = \tfrac{1}{4}x^4 + \tfrac{1}{2}ux^2 + vx.$$

The induced map $\varphi : R \times m/J \to m$ is given by

$$\begin{aligned}\varphi(x;u,v)\xi &= \xi F' + \tfrac{1}{2}\xi^2 F'' + \ldots \\ &= \xi(x^3+ux+v) + \tfrac{1}{2}\xi^2(3x^2+u) + \xi^3 x + \tfrac{1}{4}\xi^4.\end{aligned}$$

The composition $\theta = R \times m/J \to m/mJ$ is given by

$$\theta(x,u,v)\xi = \xi(x^3+ux+v) + \tfrac{1}{2}\xi^2(3x^2+u) + \xi^3 x,$$

where ξ, ξ^2, ξ^3 are reinterpreted as 3-jets rather than germs. The restriction $\theta|M : M \to m^2/mJ$ is given by

$$(\theta|M)(x;u,v)\xi = \tfrac{1}{2}\xi^2(3x^2+u) + \xi^3 x.$$

Choose for m^2/mJ the base $\{\xi^2, \xi^3\}$ and coordinates (a,b). Then $\theta|M$ is given by

$$\begin{cases} a = \tfrac{1}{2}(3x^2+u) \\ b = x \end{cases}$$

Therefore $(\theta|M)^{-1}$ is given by

$$\begin{cases} x = b \\ u = 2a - 3x^2 = 2a - 3b^2 \\ v = -ux - x^3 = -2ab + 2b^3 \end{cases} .$$

Therefore the catastrophe map $\chi: m^2/mJ \to m/J$ is given by

$$\begin{cases} u = 2a - 3b^2 \\ v = -2ab + 2b^3 \end{cases} .$$

This has Jacobian

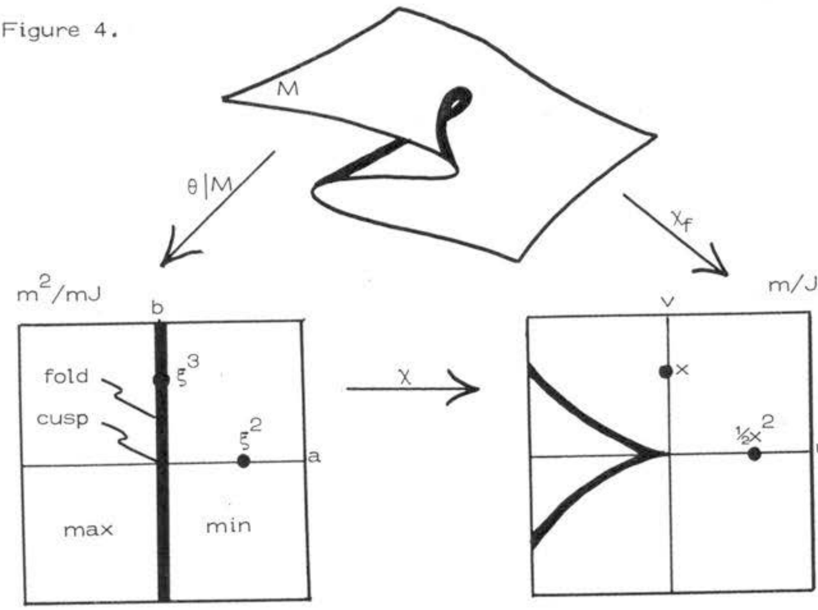
$$\frac{\partial(u,v)}{\partial(a,b)} = \begin{vmatrix} 2 & -6b \\ -2b & -2a+6b^2 \end{vmatrix} = -4a.$$

Therefore sing χ is the b-axis, $a = 0$. The canonical stratification N of m^2/m^4 comprises 4 strata :

minima	$a > 0$
maxima	$a < 0$
fold	$a = 0,\ b \neq 0$
cusp*	$a = b = 0.$

The only indeterminate stratum is the last (which is why we have starred the word cusp), because codim $N^* = k - 2 = 2$, and so dim $N^* = 0$. Therefore since $\pi^{-1}N = N$, no refinement is possible. Therefore strat $\chi = N$. In Figure 4 the stratifications are shown by thick lines.

Figure 4.



Example 2. The cuspoid, A_k.

In Arnold's notation [2] the kth cuspoid is A_{k+1}, with germ x^{k+2}, and codimension k. As in the cusp, strat $\chi = N$ and the stratification is given by a flag of linear subspaces, with one stratum in each odd, and two strata in each even codimension (except when k is even the origin is the only stratum of codimension k). This completes the case $n = 1$, and we now pass to $n > 1$.

Homogeneous forms. The jet space m^k/m^{k+1} can be identified with the space of real homogeneous forms of degree k in n variables. The $\mathcal{G}$-action in this case reduces G-action, where $G = GL(n,R)$ denotes the general linear group, because the non-linear action of $\mathcal{G}$ is quotiented out. Therefore the canonical stratification of m^k/m^{k+1} coincides with that induced by G. In particular when $n = 2$, $k = 3$ it is determined by the umbilic bracelet - which is why we proved Theorem 1. We now turn to the simpler case $n = 2$, $k = 2$ of quadratic forms in two variables :

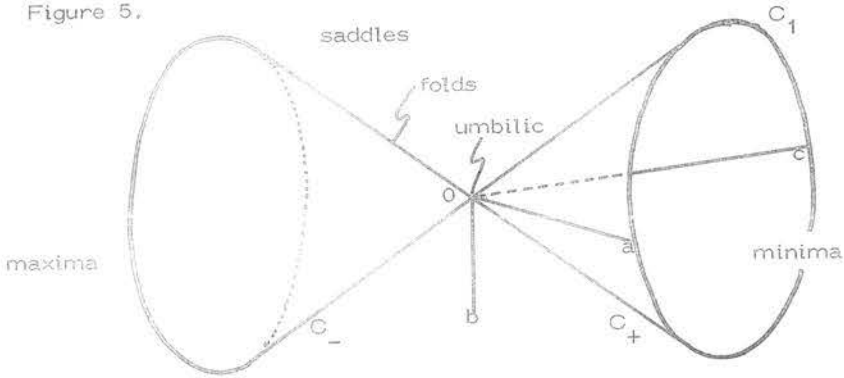
$$q = ax^2 + 2bxy + cy^2.$$

Real quadratic forms are classified linearly by rank and signature, and therefore the stratification is determined by the discriminant cone C, given by $ac = b^2$. We can therefore state without proof :

Lemma 10. When $n = 2$ the canonical stratification N of quadratic forms m^2/m^3 has 6 strata, with indeterminate subspace N^* equal to the discriminant cone C.

Corank	Name	Dim	Example	Formula
0	minima	3	$x^2 + y^2$	$ac > b^2$, $a > 0$
	maxima	3	$-x^2 - y^2$	$ac > b^2$, $a < 0$
	saddles	3	$x^2 - y^2$	$ac < b^2$
1	folds* C_+	2	x^2	$ab = b^2$, $a + c > 0$
	C_-	2	$-x^2$	$ac = b^2$, $a + c < 0$
2	umbilic*	0	0	$a = b = c = 0$

Figure 5.



<u>Definition of attaching map.</u> Subsequently we shall need to describe how strata are attached to one another, and Figure 5 presents an opportunity to introduce and illustrate a useful definition. Suppose X, Y are strata (or disjoint unions of strata) in a manifold Z, such that $X \supset Y$. We say the map $\Psi: R \times M \to Z$ <u>attaches</u> X onto Y if $\Psi(0 \times M) = Y$ and Ψ maps the complement diffeomorphically onto X. Let $M_\delta = \Psi(\delta \times M)$. Intuitively we think of M_δ moving isotopically through X, tracing out the whole of X, as δ runs through non-zero values of R, and then crushing down onto Y as $\delta \to 0$. Given an open subset $V \subset Y$, we say Ψ <u>covers</u> V n times if $\Psi|\Psi^{-1}V : \Psi^{-1}V \to V$ is an n-fold covering. We say Ψ has <u>singularities at</u> $\Psi(\text{sing } \Psi)$, which, from the

definition, is a closed subset of Y.

Example. In Figure 5 let C_1 denote the ellipse given by the intersection of C with the plane $a + c = 2$. Define $\Psi: R \times C_1 \to R^3$ by $\Psi(\delta, q) = \delta q$. Then Ψ attaches the fold strata $C_+ \cup C_-$ onto 0, with a singularity at 0. Other examples are given in Lemma 12 and Theorem 3 below.

Unfolding the umbilics. The stratification N in Lemma 10 and Figure 5 is the one that is refined by the hyperbolic, elliptic and parabolic umbilics (and also indirectly by the double-cusp) as we shall show below. In each case the refinement is non-trivial. Now there are many possible choices of unfolding, and different applications give rise to different choices and different (although diffeomorphic) bifurcation sets (see for example [14, Figure 6]). In order to best compare our formulae with those of Thom, we use his choice of unfolding for the parabolic umbilic [11, page 84]. Then, in order to best reveal the relationship between that and the other two we choose germs and unfoldings for the latter that are different to his. These give slightly different bifurcation sets, but yield simple formulae for χ.

Example 3. The hyperbolic umbilic D_4^+.

Choose the germ $f = x^2y + \frac{1}{3}y^3$. Here $n = 2$, $k = 3$, $mJ = m^3$, and therefore $\sigma = 0$, codim $f = 3$. Choose for m/J the base $\{x^2, -x, -y\}$ and coordinates (t,u,v). Choose ε by reinterpreting the base jets as germs. Therefore the unfolding is

$$F = x^2y + \tfrac{1}{3}y^3 + tx^2 - ux - vy .$$

Therefore

$$(\theta|M)(x,y;t,u,v)(\xi,\eta) = \xi^2(y+t) + 2\xi\eta x + \eta^2 y .$$

Choose for m^2/mJ the base $\{\xi^2, 2\xi\eta, \eta^2\}$ and coordinates (a,b,c). Therefore χ is given by

$$\begin{cases} t = a - c \\ u = 2ab \\ v = b^2 + c^2 \end{cases}$$

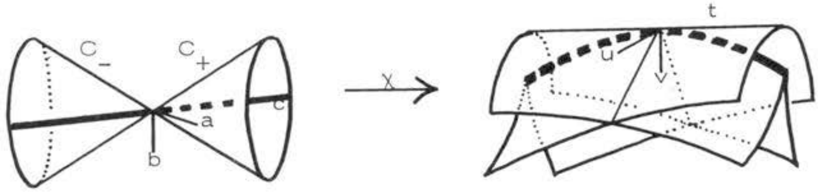
This has Jacobian $4(ac - b^2)$, confirming that

$$\text{sing } \chi = C,$$

the discriminant cone of Lemma 10. We now want to compute how strat χ refines N. Since $N^* = C$, the only strata to be refined are the two fold strata C_+ and C_-. The singularities of $\chi|C$ are given by $a^2 + 3b^2 = 0$. (This can be found by substituting $ac = b^2$ and computing where the Jacobian matrix drops in rank, or by Lagrange's method of undetermined multipliers). Hence $a = b = 0$. Therefore $\chi|C$ is singular along the c-axis, which is a generator of the cone C. This generator is separated by the origin into two half-lines. Therefore each of the two indeterminate strata, C_+ and C_-, is refined into two substrata, one substratum comprising a half-line of cusps, and the other comprising the complementary surface of folds. Therefore altogether strat χ has 8 strata.

The generator is mapped by χ into the parabola $u = 0$, $v = t^2$ which is the cusped edge of the bifurcation set.

Figure 6.



Example 4. The elliptic umbilic D_4^-.

Choose the germ $f = x^2y - \frac{1}{3}y^3$, and, apart from this one change of sign, exactly the same unfolding as the previous example :

$$F = x^2y - \tfrac{1}{3}y^3 + tx^2 - ux - vy .$$

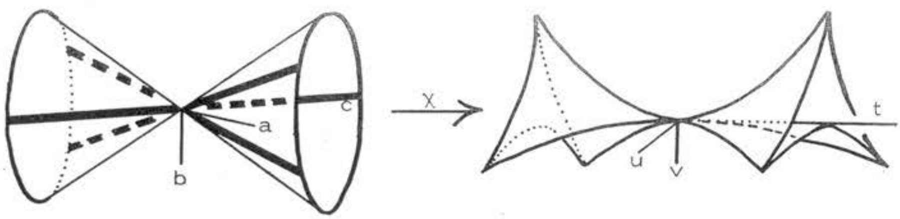
Then χ is given by

$$\begin{cases} t = a + c \\ u = 2ab \\ v = b^2 - c^2 \end{cases} .$$

This time the singularities of $\chi|C$ are given by $a^2 - 3b^2 = 0$, which gives 3 generators of C, namely the c-axis and the lines with direction ratios $(a,b,c) = (3, \pm\sqrt{3}, 1)$. Therefore each of the two indeterminate strata is refined into two substrata, one substratum comprising 3 half-lines of cusps, and the other comprising the complementary 3 components of folds. Again strat χ has altogether 8 strata (only this time they are not all connected).

Each of the 3 generators is mapped by χ into a parabola touching the t-axis, and the sections of the bifurcation set perpendicular to the t-axis are triangular hypocycloids.

Figure 7.



Example 5. The parabolic umbilic D_5.

Choose the germ $f = x^2y + \frac{1}{4}y^4$. Here $n = 2$, $k = 3$, $\sigma = 1$ because $y^3 \notin mJ$. Therefore codim $f = 4$ and $\pi^{-1}N = N \times R$. Following Thom [11 ,p. 84] choose for m/J the base $\{x^2, y^2, -x, -y\}$ and coordinates (t,w,u,v). Therefore the unfolding is

$$F = x^2y + \tfrac{1}{4}y^4 + tx^2 + wy^2 - ux - vy.$$

Therefore

$$(\theta|M)(x,y;t,u,v,w)(\xi,\eta) = \xi^2(y+t) + 2\xi\eta x + \eta^2(\tfrac{3}{2}y^2+w) + \eta^3 y.$$

Choose for m^2/mJ the base $\{\xi^2, 2\xi\eta, \eta^2, \eta^3\}$ and coordinates (a,b,c,d). Then χ is given by

$$\begin{cases} t = a - d \\ u = 2ab \\ v = b^2 + 2cd - 2d^3 \\ w = c - \frac{3}{2}d^2 \end{cases}.$$

Again this has Jacobian $4(ac-b^2)$, confirming that

$$\text{sing } \chi = C \times R$$

where C is the discriminant cone of Lemma 10, and $0 \times R$ is the d-axis. This time there are 3 strata of $N \times R$ to be refined, namely $0 \times R$, $C_+ \times R$, $C_- \times R$. The umbilic stratum $0 \times R$ is refined in 3 substrata

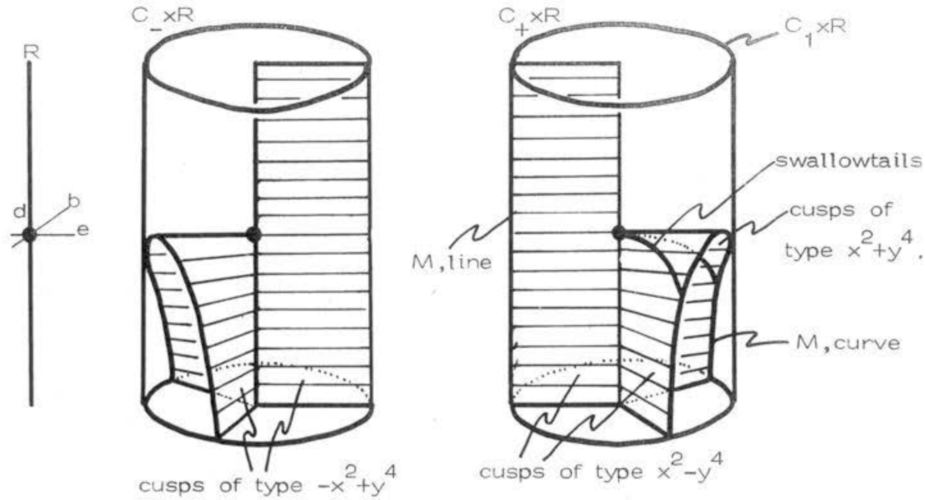
hyperbolic umbilics	$d > 0$
parabolic umbilic	$d = 0$
elliptic umbilics	$d < 0$.

Meanwhile the orther two strats $C_\pm \times R$ are refined by the formulae (which can be found by computing successive singularities of $\chi|C \times R$) :

folds	$ac = b^2 \neq -a^2d$
cusps	$ac = b^2 = -a^2d$, $a^2 \neq 4c$
swallowtails	$ac = b^2 = -a^2d$, $a^2 = 4c$, $a \neq 0$.

We can draw pictures of the refinements by squashing each end of the cone flat; more precisely the projection $R^4 \to R^3$ given by $(a,b,c,d) \to (e,b,d)$, where $e = \frac{1}{2}(a-c)$, maps each of $C_{\pm} \times R$ diffeomorphically into R^3. Figure 8 shows the images of the refinements.

Figure 8.



Note that the swallowtails only appear in $C_+ \times R$ because $a^3 = 4ac = 4b^2 > 0$, and therefore $a > 0$. Therefore $C_- \times R$ is refined into only 2 substrata, namely one 2-dimensional substratum of cusps of type $-x^2 + y^4$, with 2 components given by

$$a = b = 0, \; c < 0$$

$$ac = b^2 = -a^2 d, \; a < 0;$$

and the complementary 3-dimensional substratum of folds of type $-x^2 + y^3$ (also with 2 components). Meanwhile $C_+ \times R$ is refined into 4 substrata, namely one 1-dimensional substratum of swallowtails of type $x^2 + y^5$ (with 2 components); one connected 2-dimensional substratum of cusps of type $x^2 + y^4$, given by $a^2 > 4c$; another 2-dimensional substratum of (dual)-cusps of type $x^2 - y^4$, given by $a^2 < 4c$ (with 3 components); and the complementary

3-dimensional substratum of folds of type $-x^2 + y^3$ (with 2 components).
Summarising, we have shown :

Lemma 11. The catastrophe map of the parabolic umbilic has 12 strata as follows :

NxR	strata χ
minima	minima
maxima	maxima
saddles	saddles
$C_+ \times R$	folds, cusps, dual-cusps, swallowtails
$C_- \times R$	folds, cusps
$0 \times R$	hyperbolic, elliptic, parabolic umbilics.

Note that the bifurcation set is much more complicated because of self-intersections (see [6,11,17]).

Before leaving the parabolic umbilic we relate it to the previous examples. Notice that Cxd meets the cusp strata in 1, 2 or 3 generators according as $d \gtreqless 0$, as follows :

$d > 0$: 1 generator with direction ratios $(0,0,1,0)$

$d = 0$: 2 generators with direction ratios $(0,0,1,0),(1,0,0,0)$.

$d < 0$: 3 generators with direction ratios $(0,0,1,0),(1,\pm\sqrt{-d},-d,0)$.

These correspond to Figures 6 and 7 because points on 0xR represent hyperbolic or elliptic umbilics according as $d \gtrless 0$.

In Figure 8, 0xR is projected onto the vertical axis of each cylinder, and Cxd is projected onto the two horizontal sections at level d. Figure 8 illustrates how the hyperbolic stratum $d > 0$ lies locally in the closure of 1 sheet of cusps, whereas the elliptic stratum $d < 0$ lies in 3 sheets. Figure 8 also shows how the 3 generators merge smoothly into 1 by coalescing the other 2 at the parabolic point. We shall now rephrase these observations in terms of an attaching map, in order to furnish intuition for the analogous 7-dimensional result for the double-cusp (see Theorem 3 below).

Let $C_1 \times R$, M denote the intersection of the plane a + c = 2 with $C_+ \times R$, the cusp and swallowtail strata. In Figure 8, $C_1 \times R$ projects onto the cylinder of radius 1, and therefore M consists of a line and a curve; the line has equations $e + 1 = b = 0$, and the curve can be written parametrically $(e,b,d) = (\cos\theta, \sin\theta, -\tan^2\theta/2)$, $-\pi < \theta < \pi$. Define $\Psi: R \times M \to R^4$ by $\Psi(\delta,(q,d)) = (\delta q, d)$.

Lemma 12. Ψ attaches the cusp and swallowtail strata onto the umbilic strata, covering the hyperbolic stratum once, the elliptic stratum thrice, with a singularity at the parabolic point.

Proof. The line maps diffeomorphically onto $0 \times R$, and the curve is folded onto the elliptic stratum.

SECTION 3. THE DOUBLE CUSP.

The double cusp has germ $f = x^4 + y^4$, and belongs to the family X_9 in Arnold's notation [3,4]. Here $n = 2$, $k = 4$ and therefore $\dim(m^2/m^4) = 7$. Meanwhile $\sigma = 1$ because $x^2y^2 \notin mJ$. Therefore codim $f = 8$. Therefore the problem of finding strat χ is reduced to finding

(i) the 7-dimensional stratification N, and

(ii) the 8-dimensional refinement of N×R.

However Looijenga [8] has shown that the last factor is trivial in the sense that there is a homeomorphism

$$\text{strat } \chi \cong (\text{strat } \chi')\times R$$

where χ' is the semi-universal unfolding

$$\chi' : m^2/m^4 \to m/J+m^4$$

defined in the same way as χ, only with mJ, J replaced by $m^4, J+m^4$. Therefore problem (ii) is reduced to finding

(iii) the 7-dimensional refinement strat χ' of N.

It is possible to list the strata and their incidence relations, and to write down equations for them as in the above examples. But to my mind the problem is not satisfactorily "solved" until one achieves a more global geometric description of the way the strata fit together, that one can somehow "visualise". In this sense we shall give a solution to problem (i), but as yet I have not been able to solve problems (ii) and (iii). So let us tackle problem (i).

We can decompose m^2/m^4 by the $\mathcal{G}$-invariant short exact sequence

$$m^3/m^4 \xrightarrow{\subset} m^2/m^4 \xrightarrow{p} m^2/m^3 ,$$

where p is the projection. In the case $n = 1$ this is easy to visualise, because it is just the left-hand plane pictured in Figure 4, with

$$\text{b-axis} \xrightarrow{\subset} R^2 \xrightarrow{p} \text{a-axis}.$$

However in the case $n = 2$ we are considering, the dimensions make visualisation more difficult :

$$R^4 \xrightarrow{\subset} R^7 \xrightarrow{p} R^3$$

However we have already done the two ends, because the stratification $N^{3,4}$ of the left-hand end R^4 is given by the cubic discriminant D, or the umbilic bracelet of Theorem 1, while the stratification $N^{2,3}$ of the right hand end R^3 is given by the quadratic discriminant cone C of Lemma 10. Since p commutes with the action of $\mathcal{G}$, the stratification $N = N^{2,4}$ of the middle R^7 that we are seeking is a refinement of $p^{-1}N^{2,3}$.

Product structure. It is convenient to choose a product structure $R^7 = R^3 \times R^4$ compatible with p. The easiest way to do this is to choose coordinates x,y. Then a 3-jet $f \in R^7$ can be written $f = pf + y$ by Taylor expansion, where $pf \in R^3$ is the unique 2-jet, and $y \in R^4$ the third order term, which is determined by, but depends upon, the choice of coordinates. Although the product structure depends upon choice, the constructions that we make below are $\mathcal{G}$-invariant, and hence independent of choice. For example the stratification $N^{2,3} \times R^4 = p^{-1}N^{2,3}$ is $\mathcal{G}$-invariant.

Lemma 13. In the double-cusp N contains 12 strata as follows :

Corank	$N^{2,3} \times R^4$	$N = N^{2,4}$	dim	strat χ'
0	minima	minima	7	A_1
	maxima	maxima	7	A_1
	saddles	saddles	7	A_1

Corank	$N^{2,3} \times R^4$	$N = N^{2,4}$	dim	strat χ'
1	$C_+ \times R^4$	T^6_+, folds	6	A_2
		M^5_+, cusps*	5	A_3, A_4, A_5, A_6, A_7
	$C_- \times R^4$	T^6_-, folds	6	A_2
		M^5_-, cusps*	5	A_3, A_4, A_5, A_6, A_7
2	$0 \times R^4$	H, hyperbolic	4	D_4
		E, elliptic	4	D_4
		P, parabolic*	3	D_5
		X, exceptional*	2	E_6
		0, double cusp*	0	X_9.

Remark. The asterisks denote the indeterminate strata. The notation for the strata of N refer to Lemma 1 above and Lemma 14 below. The last column lists in Arnold's notation [2,3,4] the substrata that occur in the refinement strat χ' of N. One can show that these, and only these, substrata occur by the methods of A'Campo [1]. The substrata occur with multiplicities; for example all four substrata of cusps occur, $\pm x^2 \pm y^4$, two in each A_3. Although N is independent of the germ x^4+y^4, the list of substrata depends upon the germ. In the case of non-compact germs of X_9, namely x^4-y^4 and $x^2+y^2-6x^2y^2$, different substrata occur; for example A_7 disappears, while D_6 appears in P, and E_7 appears in X. However I do not know the geometry of all the substrata.

Proof of Lemma 13. As we have already observed, the $\mathcal{G}$-action on $0 \times R^4$ reduces to linear action, and hence the refinement in N is given by the umbilic bracelet. There remains to check $C_{\pm}\times R^4$. Any 2-jet in C_+ is equivalent to x^2, and therefore any 3-jet in $C_+\times R^4$ is equivalent to $x^2 + y^3$ or x^2, giving a determinate substratum T^6_+ of folds, and an indeterminate substratum M^5_+ of cusps (indeterminate, because the 3-jet cannot distinguish between cusps, swallowtails, etc.). Similarly for C_-. This completes the proof, and we now look at these two substrata more closely.

Lemma 14. M^5_+ is a 5-dimensional Möbius strip and T^6_+ is a 6-dimensional solid torus.

Proof. Let $q_0 = x^2 \in C_+$, and let $f \in q_0 \times R^4$. Then

$$\begin{aligned} f &= x^2 + ax^3 + bx^2y + cxy^2 + dy^3 \\ &= (x + \tfrac{1}{2}(ax^2 + bxy + cy^2))^2 + dy^3 \quad (\text{modulo } m^4) \\ &\sim x^2 + dy^3. \end{aligned}$$

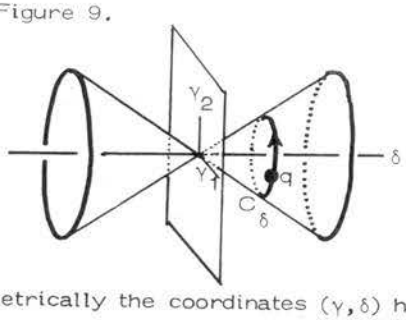
Therefore $f \in M^5_+$ if and only if $d = 0$. Therefore

$$M^5_+ \cap (q_0\times R^4) = q_0 \times R^3_0$$

where $R^3_0 \subset R^4$ is the linear subspace given by $d = 0$.

Analogous to Lemma 3, we can write the generic quadratic form using the complex variable $z = x + iy$, one complex coefficient $\gamma = \gamma_1 + i\gamma_2$, and one real coefficient δ, as follows :

Figure 9.



$$\begin{aligned} q = (\gamma,\delta) &= \mathcal{R}(\gamma z^2 + \delta z\bar{z}) \\ &= \gamma_1(x^2-y^2) - 2\gamma_2 xy + \delta(x^2+y^2) \\ &= (\delta+\gamma_1)x^2 - 2\gamma_2 xy + (\delta-\gamma_1)y^2 \end{aligned}$$

Therefore the discriminant cone C is given by $|\gamma| = |\delta|$, and $C_{\pm}$ given by $\delta \gtrless 0$.

Metrically the coordinates (γ,δ) have the attraction that C is now a circular cone with axis the δ-axis, whereas it was only an elliptical cone in the (a,b,c) coordinates of Lemma 10. Let C_δ denote the circle on C given by

δ = constant $\neq 0$. As in Lemma 4 the action of the circle group S^1 is given by

$$qg_\theta = (\gamma,\delta)g_\theta = (\gamma e^{2i\theta},\delta).$$

In other words the circle group spins C twice around its axis, leaving the circles C_δ invariant. Now let

$$q = (\delta,\delta) = 2\delta x^2 \in C_\delta \ .$$

The first half of S^1 isotopes q once round C_δ back to itself. Meanwhile, by Lemma 4, S^1 acts on R^4 by $(\alpha,\beta)g_\theta = (\alpha e^{3i\theta},\beta e^{i\theta})$. Therefore when $\theta = \pi$, $g\pi$ maps R^4 antipodally, and hence maps R^3_0 onto itself reversing orientation. Therefore the first half of S^1 isotopes $q\times R^3_0$ over C_δ through $C_\delta\times R^4$ back onto itself with orientation reversal, tracing out a 4-dimensional Möbius strip M^4_δ. Therefore

$$M^5_+ \cap (C_\delta\times R^4) = M^4_\delta$$

because N is $\mathcal{G}$-invariant, and hence S^1-invariant. Finally we show M^5_+ is a Möbius strip by scalar multiplication, as follows. Let $R^3_\theta = R^3_0 g_\theta$. Make a standard copy of the 4-dimensional Möbius strip by defining

$$M^4 = \bigcup\{e^{2i\theta}\times R^3_\theta;\ 0 \le \theta < \pi\} \subset S^1\times R^4.$$

Define

$$\Psi : R\times M^4 \to R^3\times R^4$$

$$(\delta,(e^{2i\theta},y)) \to ((\delta e^{2i\theta},\delta),y)\ .$$

Then $\Psi(\delta\times M^4) = M^4_\delta$, and Ψ maps $R_+\times M^4$ diffeomorphically onto M^5_+, proving the latter is a 5-dimensional Möbius strip.

Meanwhile the complement of $q\times R^4_0$ in $q\times R^4$ is a pair of 4-cells, which the first half of S^1 isotops onto each other preserving orientation, and forming a 5-dimensional solid torus. Scalar multiplication by R_+ gives the 6-dimensional solid torus T^6_+ . This completes the proof of Lemma 14.

To complete the description of N we need to show how the two Möbius strips are glued onto the umbilic bracelet.

Theorem 3. Ψ attaches the cusp strata $M^5_{\pm}$ onto the umbilic strata $0{\times}R^4$, covering the hyperbolic stratum H once, the elliptic stratum E thrice, and with singularities at D.

Remarks. The wording of the theorem resembles that of Lemma 12, and so some intuition can be extracted from Figure 8. A corollary of the theorem is that the induced stratification on R^3xy is isomorphic to that of the hyperbolic umbilic in Figure 6 if $y \in H$, or to the elliptic umbilic in Figure 7 of $y \in E$. Consequently in the refinement strat χ' of N, the strat substrata A_4, A_5, A_6, A_7 of swallowtails, etc. do not neet the neighbourhood of H,E and only abut D, as in Figure 8.

Proof of the theorem. Let $\psi: M^4 \to R^4$ be induced by projection $S^1{\times}R^4 \to R^4$. If we identify $M^4 = 0{\times}M^4$, $R^4 = 0{\times}R^4$ then $\psi = \Psi|M^4$. Since Ψ maps the complement of M^4 diffeomorphically onto $M^5_{\pm}$, the theorem reduces to showing ψ covers H once, E thrice, with singularities at D.

Define the core Q^3 of M^4 as follows. Recall that we defined $R^3_0 \subset R^4$ by $d = 0$; now define $R^2_0 \subset R^3_0$ by $c = d = 0$. Let $R^2_\theta = R^2_0 g_\theta$. Define

$$Q^3 = \bigcup_\theta \{e^{2i\theta} {\times} R^2_\theta;\ 0 \le \theta < \pi\} \subset M^4 .$$

Then Q^3 is a 3-dimensional solid torus, since the antipodal map of R^2_0 is orientation preserving.

Lemma 15. $\text{Sing}\ \psi = Q^3$.

$$\psi(\text{sing}\ \psi) = \psi Q^3 = \bigcup R^2_\theta = D.$$

Proof. Since ψ embeds each fibre, $\psi(e^{2i\theta} \times R^3_\theta) = R^3_\theta$,

$$\psi(\text{sing } \psi) \cap R^3_\theta = \lim_{\varphi \to \theta}(R^3_\varphi \cap R^3_\theta) = R^2_\theta \ .$$

Therefore $\psi(\text{sing } \psi) = \bigcup R^2_\theta$, and sing $\psi = Q^3$. The condition for f to lie in some R^2_θ is that f has at least two roots equal, which is the same condition for $f \in D$. This completes the proof of Lemma 15.

Projective notation. The projective point of view lends some insight, because it shows that the exceptional stratum determines both the bracelet and the Möbius strips. Let P^3 denote the 3-dimensional real projective space of lines through the origin in R^4. Then $R^2_\theta \subset R^3_\theta \subset R^4$ induce projective subspaces $P^1_\theta \subset P^2_\theta \subset P^3$. By Lemma 2, $X \subset D \subset R^4$ induces $\overline{X} \subset \overline{D} \subset P^3$.

Lemma 16. $\overline{X}$ is a twisted cubic curve with tangents $\{P^1_\theta\}$ and osculating planes $\{P^2_\theta\}$. $\overline{D}$ is the ruled surface $\bigcup P^1_\theta$ and the envelope of $\{P^2_\theta\}$.

Proof. The second sentence is a corollary of Lemma 15. $\overline{X}$ is a twisted cubic because it can be parametrised $[a,b,c,d] = [\lambda^3, 3\lambda^2\mu, 3\lambda\mu^2, \mu^3]$. The rest follows from the fact that the generators of a cubic developable are the tangents, and its tangent planes are the oscilating planes, of its edge of regression [15].

Having dealt with the singularities of ψ, to complete the proof of Theorem 3 we must now deal with its regularities.

Lemma 17. ψ covers H once and E thrice.

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