# PRESIDENTIAL ADDRESS

# ON THE CLASSIFICATION OF DYNAMICAL SYSTEMS

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I have always found classification theorems to be amongst the most deeply satisfying results in mathematics – for instance the classification of surfaces, or knots, or simple groups, or elementary catastrophes. And classification problems can often generate most fertile areas of research – as illustrated for example by our sustained attempts throughout this century to try and classify 3-manifolds.

Of course classifications differ in their completeness: for example the classification of catastrophes is not yet complete like that of simple groups, in the sense that one can compile an atlas and give an algorithm for identifying specimens. But it is complete up to a certain codimension, and it has that same deeply satisfying quality of starting from an innocent looking and intuitively appealing definition, and then, using non-trivial mathematics, deriving a surprisingly finite list that you would never have dreamt of from first glance at the original definition. Indeed one of the things that led René Thom to invent catastrophe theory in the first place was his astonishment at discovering the umbilics while playing around with light caustics: he was expecting to see the stable singularities of maps (which do not include the umbilics) and then suddenly realised that what he was looking at were in fact the stable projections of critical manifolds of parametrised functions. And the proofs, as so often in topology, employ that delicious blend of geometry and algebra, using geometry to show two things are the same, and algebra to show they are different.

But compare these nice clean branches of mathematics with the theory of differential equations. When I was a student, I confess, I cordially disliked differential equations; they seemed to be just a rag-bag of tricks rather than a theory, and they inevitably seemed to attract messy notation and arbitrary approximation. They spewed forth illogical haphazard lists of special functions, which have mostly disappeared nowadays thanks to the computer, and whose main purpose in those days seemed to be just for manufacturing tricky examination questions. It was only gradually over the years that I came to appreciate and enjoy differential equations, and to understand and tolerate my earlier prejudices against them.

So how come? If you dislike differential equations as I used to how can I persuade you to change your mind?

The first thing to do is to get to know a few famous equations personally like old friends. For example coupled linear oscillators, or the Van der Pol equation, or Duffing's, or the prey-predator, or the replicator equations, or the Euler equations of a gyroscope, or Kapitza's upside-down pendulum. Or some of the famous partial

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differential equations of physics like the wave equation, or the heat equation, or Maxwell's, or Schrödinger's, or the Korteweg–de Vries, or the Fokker–Planck. Or the Hamiltonian equations and the amazing fact that they are contained within the natural symplectic structure on the cotangent bundle of the configuration manifold. Or some nice strange attractors like the solenoid, or Lorenz's, or Hénon's, or the logistic map, or an Anosov diffeomorphism on the torus. Throw in a strange saddle like the horseshoe, and a few nice bifurcations like the cusp, the butterfly, the Hopf, some period doublings, or their Feigenbaum limit, or a homoclinic orbit, or a nonwandering explosion, and you'll be away.

I've forgotten who told me that one good example was worth ten theorems, but it's a precept that I've been trying to drum into my students for years. One nice example can often capture the quintessence of a subject, so that if you study that one example in depth you may be able to perceive the whole theory in microcosm. This is a great advantage if, like me, you are a slow reader with a poor memory, because otherwise you may never get round to mastering the general theory. I much prefer pottering to reading, pottering around particular examples, rewriting them in my own notation and thinking about them as geometric objects, maybe because I am a geometer at heart. I begin to relish them like pieces of sculpture – or perhaps a better metaphor would be like perennials in my rather untidy garden, that to my surprise and delight seem to blossom anew each season in spite of the weeds.

Some mathematicians are superstructure people: to be a superstructure person you have to be a good scholar, a good reader with a good memory, so that you can master large areas of other people's work, and then build superstructures upon their substructures. Alas not me, because I am a poor reader: I tend to rely on what I am told, and to ferret away at problems that do not require many prerequisites, and only get down to reading when I really need to know. Ignorance of course has its disadvantages, and frequently lands me in the soup when I omit to acknowledge the work of others that I really ought to have known about. On the other hand it can sometimes be an advantage, as for instance when you unwittingly reprove a known theorem, but by a new method of proof that opens up avenues that were previously unsuspected (see, for example [7, page 611]). Indeed, this is an approach that I always try to encourage amongst my research students by giving them an unsolved problem on their first day, in the hope that they might produce an entirely original line of attack, not bound by the current conventions of the subject in which they are liable to find themselves trapped as soon as they begin to study the traditional approach. It is exactly the opposite of most of my colleagues, who like to endow their students with the appropriate learning before launching them off into a research problem. But then anything you discover for yourself is usually much easier to remember than anything you read, and anything you even half-discover lends wings to the reading of the other half.

But let us return to differential equations: in suggesting one should concentrate on getting to know a few favourite examples am I not guilty of the same criticism as before, of making it into a rag-bag of a subject? Surely one must develop the general theory alongside the examples? But here we run up against an intrinsic difficulty, that the closer one gets to applications the more messy the mathematics may have to become. Applications tend to resist classification, and have a way of introducing their own types of difficulty the more realistic you try and make the models. Pure mathematicians, on the other hand, are accustomed to the luxury of axiomatising away those messy difficulties; no wonder they can prove such nice clean classification theorems if they allow themselves such nice clean axioms. Indeed we can see a hint of this happening today amongst the topologists who have moved into differential equations over the last twenty years: they have hijacked the very name 'dynamical systems' away from its traditional meaning of modelling the dynamics of real physical systems, and have converted it into the study of diffeomorphisms and maps of compact manifolds. It is true that continuous flows (= solutions of differential equations) and discrete flows (= iterations of maps) do possess analogous theories, and that the discrete case does have the advantage of exhibiting chaos in one dimension lower than in the continuous case, but there are dangers in disengaging a subject from its applications and overemphasising the pure side. To take a trivial example, notice how some writers tend to forget the difference between an attractor and a repellor. For instance, what is the point of emphasising the chaos in a logistic map when it is the repellor that is chaotic while the attractor is periodic? Or what is the purpose of calculating the entropy of a horseshoe, when it is really a strange saddle and will never be observed in the attractors or their basins of attraction?

On the other hand, the advantage of the recent invasion of topology into differential equations is the boost that it gives to the general theory, enabling us to look at the classical examples with fresh eyes and recognise anew the genius for those who first found them. Another advantage is the emphasis that it places upon the global viewpoint, in considering for instance the phase portrait of *all* orbits, and the space of *all* differential equations or *all* vector fields on a manifold. In fact topology and differential equations have had a very interesting love-hate relationship with one another throughout this century, each withdrawing from the other from time to time for a period of separate evolution, only to come together again to nurture one another, because they both need each other. Without topology differential equations is liable to be enmeshed in complexity and to lose sight of the underlying simplicity, and without differential equations topology is liable to drift into abstraction and to lose touch with reality and the rest of mathematics, until it begins to crumble under its own weight.

Indeed, Poincaré invented topology in the first place to handle the horrendous analytic difficulties he encountered when founding the qualitative theory of ordinary differential equations. After the foundations of point-set topology had been established, the two subjects came together again for a brief flirtation in the 1920s with the creation of topological dynamics, but then the latter got so bogged down in the pathology of the fine structure of orbits in two dimensions that it almost killed the relationship. So topology, after its initial analytic phase of securing the foundations, withdrew to evolve separately through a geometric phase of global understanding in the 1930s, succeeded by an algebraic phase of systematisation and computation in the 1940s and 1950s. There followed another geometric phase in the late 1950s and early 1960s, except that this time it was differentiable, and only then was topology ready to return once more, refurbished and repolished, to its parent. After all, they are *differential* equations rather than *topological* equations, and so it was only to be expected that topology should have to undergo a phase of differential evolution before it was going to be any real use in this area. And indeed, as a result, we have recently witnessed a marvellous reflowering of the qualitative theory of ordinary differential equations. The analogous development in partial differential equations is perhaps yet to come.

Under the impact of this reflowering one is emboldened to raise the question of classifying all ordinary differential equations. Of course, such a task is impossible because it is too complicated, so perhaps we should first try and classify the 'generic' ones. Since any higher order equation can be written as a first order equation on a higher dimensional manifold the problem reduces to the classification of vector fields on a manifold. So let V denote the space of smooth vector fields on a manifold X. What should a classification programme consist of? We suggest four steps:

1. Choose an equivalence relation on $V$, and define a vector field to be *stable* if it has a neighbourhood of equivalents in $V$.
2. Prove that the stables are dense in $V$.
3. Classify the stable classes.
4. Classify the unstable classes of codimension $1, 2, \ldots,$ etc.

Let me explain the programme. If we are going to classify into equivalence classes then obviously we are going to need an equivalence relation: hence Step 1. The choice of equivalence relation will depend upon its usefulness, both its meaningfulness in applications and its fruitfulness for proving theorems; too coarse a relation will lump together things that 'ought to be distinct', while too fine a relation will separate things that 'ought to be the same'. But the key criterion here will be the notion of stability, not the asymptotic stability of single orbits but the structural stability of the whole differential equation. Why is stability so important? The answer is that if we are going to use a differential equation to model some application, then the model is bound to be only an approximation, and tomorrow's experiment is bound to be a perturbation of today's, so if our model is going to be of any use in describing the application and predicting tomorrow's experiment then it must be robust under perturbation. In other words it must be stable, so that perturbations of it are equivalent to it; and if it isn't stable then we must be able to choose a perturbation that is, which is only possible in general provided the stables are dense in $V$. Hence Step 2 of our programme. Of course Step 2 is the crunch line behind Step 1: we have to choose an equivalence relation that is sufficiently fine to distinguish things that are qualitatively different, but sufficiently coarse to be able to prove Step 2. Having got over the hurdle of Step 2 we can then tackle Step 3, the business of listing the stable classes. Meanwhile the unstables, with a bit of luck, will consist of various submanifolds of $V$, admittedly of infinite dimension but more importantly of finite codimension, together with some rubbish of infinite codimension that we need not worry about. The significance of these submanifolds is that any r-parameter family of vectors fields will generically meet only those submanifolds of codimension $\leqslant r$, so that listing the latter will classify all the types of bifurcation that can occur in generic r-parameter families. Moreover a small r-disk transversal to a submanifold of codimension r will give a generic unfolding of the corresponding bifurcation. Thus Step 3 classifies the generic differential equations, while Step 4 classifies their generic bifurcations. Hence the motivation behind the programme.

But why should we be so bold as to envisage such an ambitious programme in so classical a field as differential equations? The answer is that Thom [6] has given us a prototype that works for gradient systems. Admittedly gradient systems represent only a small corner of the subject, because they only have point attractors, but they can nevertheless give some insight into the more general case; for example replace those point attractors by hyperbolic strange attractors and we obtain a picture of Smale's spectral theorem for his Axiom A diffeomorphisms and flows [5].

So let us look more closely at Thom's prototype. Let $X$ be a compact manifold and $F$ the space of smooth real functions on $X$. A gradient system is a differential equation of the form $\dot{x} = -\nabla f$, $x \in X, f \in F$. Choose the equivalence relation on $F$ as follows: define two functions, $f, f'$ to be equivalent if they are conjugate, that is to say there exist diffeomorphisms $\alpha, \beta$ such that the diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & \mathbb{R} \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} \\ X & \xrightarrow{\ f'\ } & \mathbb{R} \end{array}$$

Then the programme runs as described above: stables are dense; a stable function is a Morse function, in other words a function that has a finite number of critical points, each non-degenerate, with distinct critical values, and which is classified by its critical points. If one chooses the slightly finer definition of *local equivalence*, namely $\exists\alpha$, $\forall x \in X$, $\exists\beta$, such that the diagram commutes in a neighbourhood of $x$, then the critical values do need to be distinct, and the unstable classes of codimension $1, 2, \ldots$ are just the elementary catastrophes.

Thom's success with gradient systems raised hopes for a classification of all vector fields. The favoured equivalence relation was that of topological equivalence, giving rise to the definition of *structural stability*, which was introduced by Andronov and Pontrjagin in 1937 [1], and has been developed extensively in the Russian, American and Brazilian schools. Here two vector fields $v$, $v'$ are defined to be *topologically equivalent* if there is a homeomorphism of $X$ onto itself throwing $v$-orbits onto $v'$-orbits. There is, however, an implicit inelegance in this definition because it lies outside the smooth category; therefore, to say that some model in applied mathematics is topologically equivalent to a standard model gives no assurance of a smooth change of coordinates with respect to which the applied model is standard, in contrast to the Thom theory. So topological equivalence is of limited use in applications.

But more seriously it turned out to be too fine an equivalence relation because the structurally-stables were soon discovered to be non-dense, thus failing Step 2 and torpedoeing the whole classification programme. Attention therefore switched away from classification and towards showing that structural stability was characterised by the properties of hyperbolicity and strong transversality, which in turn began to emerge as the more important concepts [5]. Meanwhile topological equivalence is now beginning to lead the subject astray again in exactly the same way that topological dynamics led it astray in the 1920s; instead of focusing upon the more important aspects in higher dimensions the subject is once more getting bogged down in the low-dimensional pathology of the fine structure, only this time it is in three dimensions rather than two. All we seemed to have gained in sixty years is one dimension!

Lest I give the wrong impression let me hasten to add that I think it is very important to understand the geometric structure of 3-dimensional flows, but I would prefer that attention were directed towards properties of that structure that are robust under perturbation and generalisable to higher dimensions, rather than towards those aspects of the fine structure that are unstable. Certainly from the point of view of applied mathematics emphasis upon the fine structure can be misplaced, because in most applications there is usually some intrinsic lower bound $\varepsilon > 0$ representing experimental error, below which measurements are meaningless. There would seem to be no harm, therefore, in sacrificing precision up to order $\varepsilon$, and indeed it would seem to be very worthwhile if, by introducing a stochastic element of order $\varepsilon$, one could stay within the smooth category and resuscitate the classification programme. This was the motivation that lay behind a recent paper that I wrote for the first issue of *Nonlinearity* [**9**]. The idea is as follows.

Let $X$ be a manifold. Given a smooth vector field $v$ on $X$ we want to find a smooth function $u$ on $X$ that describes $v$ up to order $\varepsilon$. Intuitively we want to solve the associated differential equation $\dot{x} = v$, find the attractors, take the asymptotic measure on those attractors (largest where the dynamic lingers longest), and then $\varepsilon$-smooth that measure. That was the intuitive idea, and now for the precise definition. Let $U$ be the space of smooth probability functions on $X$, namely functions $u$: $X \to \mathbb{R}$ such that $u > 0$ and $\int u = 1$. Let $u \in U$ be the steady state, given by $\partial u/\partial t = 0$, of the Fokker–Planck equation for $v$ with $\varepsilon$-diffusion, namely

$$\frac{\partial u}{\partial t} = \varepsilon \Delta u - \nabla \cdot (uv).$$

Here the solution of the Fokker–Planck equation represents a population $u(t)$ being driven along by the vector field $v$ at the same time as being subject to $\varepsilon$-small diffusion, and eventually homing in towards the steady state $u$. In fact:

THEOREM 1. *If $X$ is compact then the steady state $u$ exists and is unique, and all solutions of the Fokker–Planck equation tend to $u$.*

We can now use the steady state $u$ as a tool to study $v$. Indeed, we have a map $\pi: V \to U$ given by $v \to u$ from the space of all vector fields $V$ to the space of functions $U$ (which is, of course, dependent upon the parameter $\varepsilon$).

THEOREM 2. *The map $\pi: V \to U$ is a smooth trivial fibration with fibre the divergence-free vector fields, and cross-section the gradient vector fields.*

We can now use $\pi$ to lift the whole of Thom's classification of functions in $U$ to give a classification of vector fields in $V$ satisfying all four steps of our classification programme. Indeed, it is a strict generalisation to all vector fields of Thom's original classification of just the gradient vector fields. In detail, Step 1 of the programme is as follows: define two vector fields $v, v'$ to be *$\varepsilon$-equivalent* if their steady states $u, u'$ are locally equivalent as functions, and define $v$ to be *$\varepsilon$-stable* if it has a neighbourhood of $\varepsilon$-equivalents. It is easy to see from Theorem 2 that a vector field $v$ is $\varepsilon$-stable if and only if its steady state $u$ is a Morse function. Steps 2, 3, and 4 follow immediately:

COROLLARY. (i) *$\varepsilon$-stable vector fields are open dense.*
(ii) *$\varepsilon$-stable vector fields are classified by Morse functions.*
(iii) *$\varepsilon$-unstable vector fields of codimension $r$ are classified by the elementary catastrophes of codimension $r$.*

Theorem 2 is joint work with Santiago López de Medrano and Marc Chaperon, and the analogous result for diffeomorphisms is joint work also with Charlotte Watts [**2, 4**]. The proof of Theorem 1 uses a Perron–Frobenius type argument: the time-$t$ map of the evolution flow is a compact strongly positive operator, and the resulting unique eigenvector in $U$ turns out to be the steady state. In the proof of Theorem 2 one first shows that $\pi$ is smooth, and then uses $\pi$ to construct the cross-section and show that the fibration is trivial. A fibration is in particular an open map, and therefore, since $\pi$ is both continuous and open, we can lift the whole classification structure from $U$ to $V$.

One can further tinker about by letting $\varepsilon \to 0$, and define a notion of stability that is independent of $\varepsilon$, for which the stables are still dense [9]. And in special cases one can take the limit of $u$ as $\varepsilon \to 0$, but this takes us out of the smooth category and away from the general case, so it is perhaps drifting away from our main theme. Also one can show that as $\varepsilon \to \infty$ the fibration tends to the classical Hodge decomposition of $V$. But rather than pursue the general theory let me now return to a few examples, which, if my earlier precept is to be believed, ought to be better value than any more theorems.

EXAMPLE 1. Let $X = \mathbb{R}$ and $v = -x$, where $x \in X$.

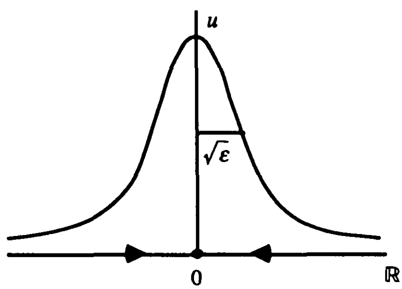


FIG. 1. The steady state of a point attractor.

The associated differential equation $\dot{x} = -x$ has a point attractor at the origin 0. It is easy to solve the Fokker–Planck equation in this case and the steady state turns out to be the normal distribution with mean 0 and variance $\varepsilon$. Any initial distribution is driven by the vector field towards the attractor but never quite gets there because of the diffusion, and so instead approaches the normal distribution, which represents the unique balance between attraction and diffusion.

If $\varepsilon \to 0$ the steady state tends to the Dirax $\delta$-function at 0, which is the asymptotic measure on the attractor. Since the normal distribution is an $\varepsilon$-smoothing of the Dirac measure, the steady state is indeed an $\varepsilon$-smoothing of the asymptotic measure. Inasmuch as all differential equations are a generalisation of this one, the concept of the Fokker–Planck steady state can be regarded as a generalisation of the normal distribution.

EXAMPLE 2. Let $X$ be compact, and let $v$ be the gradient vector field $v = -\nabla f$, where $f: X \to \mathbb{R}$. Then, by substitution into the Fokker–Planck equation, it is easy to show that the steady state is

$$u = Ke^{-f/\varepsilon}, \text{ where } K = \text{constant}.$$

Therefore $u$ has a maximum above each minimum of $f$, and vice versa. Hence

$$f \sim f' \Leftrightarrow u \sim u' \Leftrightarrow v \sim v'.$$

Consequently our equivalence relation on gradient vector fields is the same as Thom's, verifying that our theory is a strict generalisation of his.

EXAMPLE 3. The next two examples have cyclic attractors rather than point attractors. First consider the *mock Van der Pol equation* in the plane given by

$$\dot{r} = r(1-r), \quad \dot{\theta} = 2 - r\cos\theta,$$

where $(r, \theta)$ are polar coordinates. Here the unit circle $C$ is an attractor, the origin is a repellor, and all the other orbits flow towards $C$. On the attractor itself the flow is slowest where $\theta = 0$ and fastest where $\theta = \pi$. Therefore the steady state $u$ resembles a volcano crater with the rim of the crater above the attractor, a maximum above $\theta = 0$, a saddle above $\theta = \pi$, and a minimum above the repellor, as shown in Figure 2.

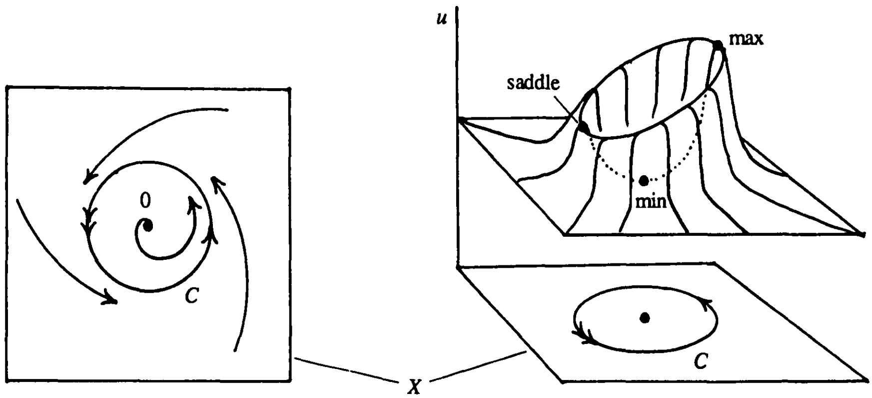


FIG. 2. The steady state of the mock Van der Pol.

The next example is the *true Van der Pol equation* given by

$$\ddot{y} + K(3y^2 - 1)\dot{y} + y = 0, \quad \text{where } K = \text{large constant}.$$

We can introduce a variable $x$ and write this as a first order equation on the plane:

$$\dot{x} = -y/K, \quad \dot{y} = K(x + y - y^3),$$

where $(x, y)$ are cartesian coordinates. To understand the geometry let $S$ denote the curve given by $x + y - y^3 = 0$. Here $S$ stands for *slow manifold* and is shown dotted in Figure 2. Since $K$ is large the vector field is large and nearly vertical away from $S$, pointing downwards at points above $S$, and upwards at points below $S$. Therefore the upper and lower branches of $S$ are attracting, while the middle branch is repelling. On $S$ itself the flow is slow, towards the left on the upper branch and towards the right on the lower branch. Therefore the whole flow has a cyclic attractor $A$ consisting of two slow legs, approximately along the upper and lower branches of $S$, joined by two fast legs, corresponding to catastrophic jumps at the fold points of $S$ onto the opposite branch. The origin is a repellor. The steady state $u$ resembles two parallel mountain ridges above the slow legs, with two maxima near the fold points of $S$, two saddles on the fast legs, and a minimum above the origin.

These two examples are topologically equivalent, but not $\varepsilon$-equivalent because the two steady states have different numbers of maxima. Engineers would be the first to agree that the equations were qualitatively different, since they would use them to model very different physical phenomena, and in this respect our new equivalence relation is much closer to applied mathematics.

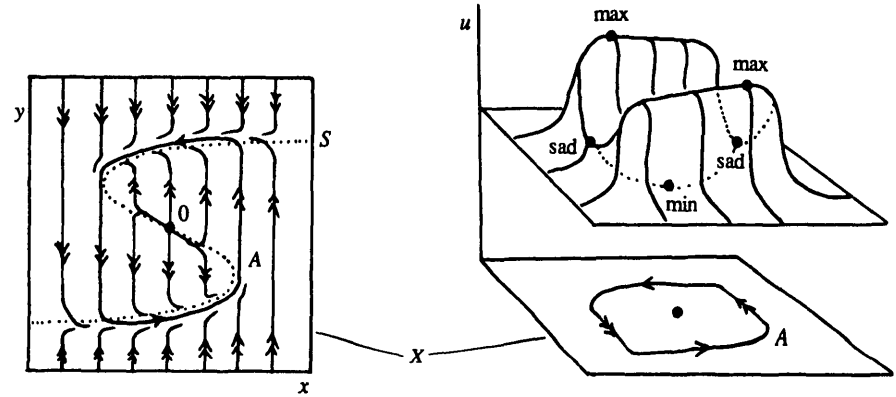


FIG. 3. The steady state of the true Van der Pol.

EXAMPLE 4. Our next example is the solenoid, which is a strange attractor. I hope the experts will forgive me if I digress a little to explain what a strange attractor is for the benefit of readers who have not met one before. If we are given a vector field $v$ on a manifold $X$ then the solution of the associated differential equation $\dot{x} = v$ is called the flow, and individual solution curves are called orbits. A subset $A$ of $X$ is called a *strange attractor* of $v$ if it is attracting, indecomposable and chaotic. Here *attracting* means that all the orbits near $A$ flow towards $A$, and none flow away from $A$. *Indecomposable* means that there is an orbit in $A$ that flows densely over $A$, so that $A$ cannot be decomposed into two pieces. *Chaotic* means that some nearby orbits in $A$ flow apart within $A$, in other words the flow on $A$ has an expanding property implying a sensitive dependence upon initial conditions, so that a small change in the latter can cause a large change in the ensuing motion. Therefore, although the flow is deterministic and predictable in the sense that it is determined by the attractor, yet it is unpredictable from the point of view of measuring coordinates on the attractor. That is the meaning of chaotic, and perhaps *chaotic attractor* would be a better name than strange attractor, since by now many of them are quite familiar. The point attractors in Examples 1 and 2 above and the cyclic attractors in Example 3 all satisfy the attracting and indecomposable properties, but are not chaotic. To construct a chaotic attractor we must focus upon the expanding property. We shall first construct a chaotic map, then perturb it to a chaotic embedding, and finally suspend it to a chaotic flow.

The simplest compact connected manifold other than a point is the circle $C = \{w \in \mathbb{C}; |w| = 1\}$, and the simplest expanding map of a circle is the double cover $f: C \to C$ given by $fw = w^2$. The map $f$ is chaotic because it is expanding, doubling the angle between nearby points. The whole circle is itself indecomposable under $f$ because for most points $w$ the discrete orbit $w, fw, f^2 w, \ldots$ induced by iterations of $f$ is dense in $C$.

Next we have to perturb $f$ to an embedding. Let $D$ denote the disk $D = \{z \in \mathbb{C}; |z| \leq 2\}$, and let $g$ denote the inclusion $C \subset D$. Then $f \times g: C \mapsto C \times D$ given by $w \mapsto (w^2, w)$ is a perturbation of $f$ that embeds $C$ inside the solid torus $C \times D$. Moreover, by shrinking the $D$ coordinate we can extend $f \times g$ to give an embedding of the whole solid torus in itself

$$e: C \times D \to C \times D, \quad e(w, z) = (w^2, w + z/4).$$

Let $T = C \times D$. Then $eT$ is a thinner solid torus that winds twice round $T$, as shown in Figure 4, and $e^2T$ is an even thinner solid torus that winds four times round $T$ inside $eT$, and so on.

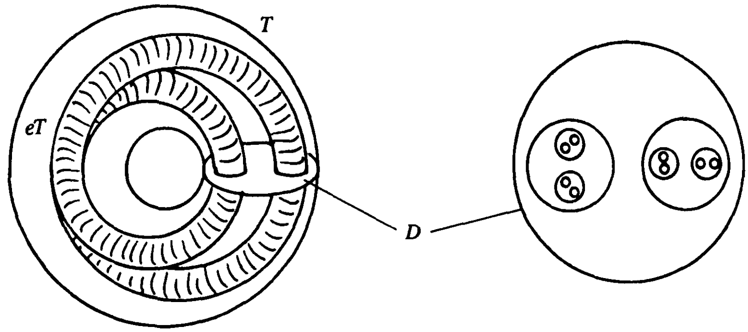


FIG. 4. The solenoid.

Let $A = \bigcap_{n>0} e^n T$. We call $A$ the *solenoid*. Consider a typical disk $w \times D$, where $w \in C$. Now $eT$ meets this disk in two small subdisks, $e^2T$ meets it in four smaller subdisks, and so on, until finally $A$ meets it in the intersection of this nest of subdisks, which is a Cantor set. Therefore $A$ is an infinite braid with Cantor cross-section.

For all $x \in T$, the discrete orbit $x, ex, e^2x, \ldots$ tends towards $A$, and so $A$ is attracting under the discrete flow generated by $e$. One can show [**8**] that $A$ contains a dense orbit and so $A$ is indecomposable. Finally, $A$ inherits its chaotic property from $f$, and so $A$ is a strange attractor of $e$. Locally $A$ is the product of a curve and a Cantor set: it is continuous in the crucial expanding direction of $C$ and Cantor in the less important contracting directions of $D$.

Our next task is to *suspend* the solenoid in order to construct a continuous flow, as follows. Let $T \times I$ be the product of $T$ with the unit interval $I$, and identify $T$ with the front end $T \times 0$. Then form $\Sigma$ from $T \times I$ by glueing the back end onto the front end using $e$, as shown in Figure 5; more precisely identify $(x, 1) = (ex, 0)$, $\forall x \in T$. Then $\Sigma$ is a 4-manifold with boundary.

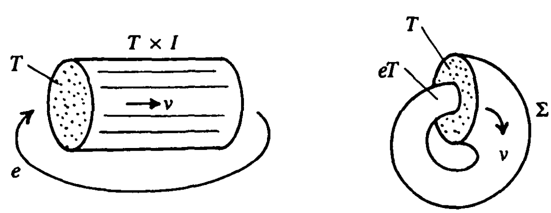


FIG. 5. The suspension of an embedding.

Let $v$ denote the vector field on $\Sigma$ induced by the unit vector field on $T \times I$ in the $I$ direction. We call $\Sigma$ the *suspension* of $T$, and we call the vector field $v$ the *suspension* of the embedding $e$. Let $\Lambda$ be the suspension of $A$, in other words the image of $A \times I$ in $\Sigma$. We shall show that $\Lambda$ is a strange attractor of $v$. To see this, notice that the continuous $v$-orbits are unions of images in $\Sigma$ of the horizontal flow lines $x \times I$ in $T \times I$. Given $x \in T$, the continuous $v$-orbit through $x$ will flow once round $\Sigma$, through $ex$, then again round $\Sigma$, through $e^2x$, and so on, flowing continuously towards $\Lambda$ in $\Sigma$ in exactly the same way that the discrete $e$-orbit $x, ex, e^2x, \ldots$ flows discretely towards $A$ in $T$. In other words, the continuous orbit is just the suspension of the discrete orbit. Therefore $\Lambda$ is attracting because $A$ was, is indecomposable because $A$ was, and is chaotic because $A$ was. Hence $\Lambda$ is a strange attractor, as required. Locally $\Lambda$ is the product of a surface and a Cantor set: it is continuous in the flow direction $I$ and the expanding direction $C$, and Cantor in the contracting direction $D$.

We want to understand what the Fokker–Planck steady state looks like for the continuous attractor $\Lambda$, but it will be simpler to visualise the corresponding steady state for the discrete attractor $A$. So let us first briefly describe the analogous theory for diffeomorphisms.

Let $X$ be a manifold and $D$ the space of diffeomorphisms $d: X \to X$. As before let $U$ denote the space of smooth probability measures on $X$, in other words smooth functions $u: X \to \mathbb{R}$ such that $u > 0$ and $\int u = 1$. Given $d \in D$, let $d^*: U \to U$ denote the induced map on measures. Let $H: U \to U$ denote the time-$\varepsilon$ map of the heat equation $\partial u/\partial t = \Delta u$, in other words, if $u_t$ denotes the solution of the heat equation arising from initial condition $u_0$, define $H(u_0) = u_\varepsilon$. In effect $H$ introduces a little bit of diffusion. Let $L$ denote the composition of the two operators, $L = d^* \circ H$.

THEOREM 3. *The map $L$ has a unique fixed point $u$, which we call the steady state of $d$, and all points of $U$ tend to $u$ under iterations of $L$.*

THEOREM 4. *The map $D \to U$ given by $d \mapsto u$ is a smooth trivial fibration with fibre the volume preserving diffeomorphisms.*

I shall leave the reader to compare the diffeomorphism theory with vector space theory (see [**2**, **4**, **9**]), because I now want to play a little trick.

Since this theory is all about measures, and since the Dirac measure at a point can be divided in half, why can't we, with our measure-theoretic spectacles on, divide the point in half? More precisely why can't we 'expand' a point 0 into two points $P$ and $Q$ by dividing the Dirac measure at 0 in half, and mapping half to $P$ and the other half to $Q$. Better still, why not 'expand' 0 into two points and then map both the points back onto 0 itself; let us call this object the 'double cover of a point'. We now have an expanding germ for manufacturing a strange attractor, just as the double cover of a circle was the expanding germ of manufacturing the solenoid.

Proceeding as before, first multiply the receiving point by $\mathbb{R}$, and then perturb the 'double cover' into a pair $d = \{d_0, d_1\}$ of embeddings $0 \to \mathbb{R}$ by defining $d_0(0) = 0$ and $d_1(0) = 2/3$. Then extend $d$ to a pair of contracting embeddings $\mathbb{R} \to \mathbb{R}$ by defining

$$d_i(x) = d_i(0) + x/3, \quad \text{where} \quad x \in \mathbb{R} \quad \text{and} \quad i = 0, 1.$$

It can be seen that $d_0$ is merely the one-third contraction towards its fixed point 0, while $d_1$ is merely the one-third contraction towards its fixed point 1. Let $d_i^*: U \to U$ be the induced maps of the smooth probability measures, and define

$$d^* = \tfrac{1}{2}(d^* + d_1^*)$$

in accordance with our initial concept of 'expanding' the point.

We claim that $d$ has a strange attractor, which is none other than the classical middle-third Cantor set. To see this, let $m$ be the characteristic measure on the unit interval $[0, 1]$. Then $d^*m$ will be the characteristic measure on $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and so on; each iteration of $d^*$ will have the effect of removing the middle third of each interval, and so the limit of the iterations will be the characteristic measure on the classical Cantor set.

We must now look for the steady state $u$ of $d$, which is the fixed point of $L = d^* \circ H$, and which will be an $\varepsilon$-smoothing of the measure on the Cantor set. It turns out that this is a smooth function with a finite number of peaks. Roughly speaking if we continue the Cantor construction of removing middle thirds until we reach a set of intervals each of which has length approximately $\sqrt{\varepsilon}$ then the peaks of $u$ will lie over those intervals. For example if $\varepsilon = \frac{1}{100}$ then $\sqrt{\varepsilon} = \frac{1}{10}$, and so $u$ will have four peaks as in Figure 6.

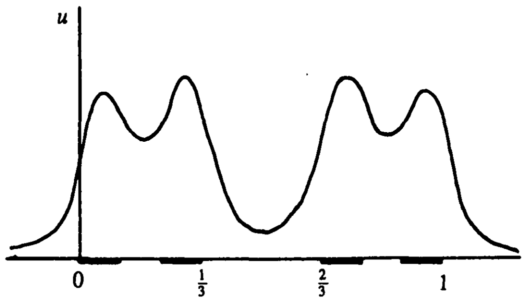


FIG. 6. The steady state of the classical Cantor set strange attractor.

The effect of the heat flow $H$ is to fuse the left pair of peaks together by diffusion, and similarly the right pair; then $\frac{1}{2}d_0^*$ maps the resulting two fused peaks back onto the left pair, while $\frac{1}{2}d_1^*$ maps them back onto the right pair. Hence $L$ keeps $u$ fixed. The important lesson to be learnt here is that all the fine structure of the Cantor set will be hidden underneath these four peaks.

We can now visualise the steady state of the solenoid. As $n$ increases the solid tori $e^n T$ become longer and thinner, and so we can choose $N$ such that $e^N T$ has thinness approximately $\sqrt{\varepsilon}$. Then $e^N T$ will meet the typical disk $w \times D$ in a set of $2^N$ little subdisks of that diameter. The section of the steady state over this disk will have $2^N$ peaks over those little subdisks, hiding the fine structure of the Cantor set $A \cap (w \times D)$ underneath. As we go round $C$ the peaks will trace out a finite braid over $e^N T$, hiding the fine structure of $A$ underneath.

The solenoid was relatively easy to understand because it is hyperbolic, that is to say, at each point there are well-defined expanding and contracting directions. Most strange attractors, however, are not so well behaved; for example the Hénon attractor [3] probably has a dense set of points where it is not hyperbolic, and its fine structure is still an intractable problem. The steady state $u$, however, hides all this fine structure under a finite number of peaks, and for most parameter values will be stable. It doesn't matter, for example, that small perturbations of the Hénon map may have an infinite number of sinks, each with its own tiny basin of attraction, because this pathology will remain hidden under the peaks, and will never be observed if there is an experimental error of order $\varepsilon$. Thus the theory may have the potential to handle aspects of strange attractors that were previously intractable.

An interesting feature of this whole approach is that it combines both measure and geometry. Usually a measure-theoretic approach is liable to destroy the geometry, because every invariant measure is equivalent to one on the unit interval. But here the geometry is preserved, and the steady state $u$ is a smooth measure on the manifold with both geometric and physical meaning, in the sense that if we programme a computer to represent orbits by dots on a print-out then $u$ will describe the density of those dots. So it has a very direct link with applications, and I look forward to seeing computer pictures of the steady states of various strange attractors if I can persuade anyone to draw them.

There is one disadvantage, however, which is shared by most computer print-outs, that although the drawing of the picture is a dynamic process, the final print-out is static, and the dynamics has been lost. It seems a pity to have to represent a dynamical system $v$ by a static picture $u$. David Epstein has made an interesting proposal, however, that it might be possible to repeat the analysis with 1-forms rather than functions, and thus incorporate the dynamic into the steady state. Alternatively one can extend the analysis from the manifold to its tangent bundle, and obtain a steady state that captures the dynamic and gives an equivalent theory.

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